# TERMINAL QUOTIENT SINGULARITIES IN DIMENSION THREE VIA VARIATION OF GIT

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Abstract. A 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ admits the economic resolution $Y \to X$, which is "close to being crepant". This paper proves that the economic resolution $Y$ is isomorphic to a distinguished component of a moduli space of certain $G$-equivariant objects using the King stability condition $\theta$ introduced by Kędzierski [11].

Contents



## 1. Introduction

The motivation of this work stems from the philosophy of the *McKay correspondence*, which says that if a finite group $G$ acts on a variety $M$, then a crepant resolution of the quotient $M/G$ can be realised as a moduli space of $G$-equivariant objects on $M$.

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Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group. A $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a *$G$-constellation* if $\mathrm{H}^0(\mathcal{F})$ is isomorphic to the regular representation $\mathbb{C}[G]$ of $G$ as a $\mathbb{C}[G]$-module. In particular, the structure sheaf of a $G$-invariant subscheme $Z \subset \mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$, which is called a *$G$-cluster*, is a $G$-constellation. Define the GIT stability parameter space

$$\Theta = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right) = 0\right\},$$

where $R(G)$ is the representation ring of $G$. For $\theta \in \Theta$, we say that a $G$-constellation $\mathcal{F}$ is $\theta$-(semi)stable if $\theta(\mathcal{G}) > 0$ ($\theta(\mathcal{G}) \geq 0$) for every nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$. A parameter $\theta$ is called *generic* if every $\theta$-semistable $G$-constellation is $\theta$-stable.

Let $\mathcal{M}_\theta$ be the moduli space of $\theta$-semistable $G$-constellations. In the celebrated paper [1], Bridgeland, King and Reid proved that for a finite subgroup $G$ of $\mathrm{SL}_3(\mathbb{C})$, $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^3/G$ if $\theta$ is generic. Craw and Ishii [2] showed that in the case of a finite abelian group $G \subset \mathrm{SL}_3(\mathbb{C})$, *any* projective crepant resolution can be realised as $\mathcal{M}_\theta$ for a suitable GIT parameter $\theta$.

While the moduli space $\mathcal{M}_\theta$ need not be irreducible [4] in general, Craw, Maclagan and Thomas [3] showed that for generic $\theta$, $\mathcal{M}_\theta$ has a unique irreducible component $Y_\theta$ containing the torus $(\mathbb{C}^\times)^n/G$ if $G$ is abelian. The component $Y_\theta$ is birational to $\mathbb{C}^n/G$ and is called the *birational component*[1] of $\mathcal{M}_\theta$.

On the other hand, in the case of $G \subset \mathrm{GL}_3(\mathbb{C})$ giving a terminal quotient singularity $X = \mathbb{C}^3/G$ in dimension 3, $X$ has the *economic resolution* $\phi\colon Y \to X$ satisfying

$$K_Y = \varphi^*(K_X) + \sum_{1 \leq i < r} \frac{i}{r} E_i$$

with $E_i$'s prime exceptional divisors. Kędzierski [11] proved that $Y$ is isomorphic to the normalization of $Y_\theta$ for some $\theta$. The main theorem of this paper is that the economic resolution $Y$ of $X$ can be interpreted as a component of a moduli space of $G$-constellations as follows.

**Theorem 1.1** (Theorem 4.19)**.** *The economic resolution $Y$ of a 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ is isomorphic to the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.*

To prove the theorem, first we generalize Nakamura's result [16]. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite diagonal group. Nakamura [16] introduced a $G$-graph which is a $\mathbb{C}$-basis of $\mathcal{O}_Z$ for a torus invariant $G$-cluster $Z$. Using $G$-graphs, he described a local chart of $G$-Hilb. In this paper, we introduce a *$G$-prebrick* which is a $\mathbb{C}$-basis of $\mathrm{H}^0(\mathcal{F}) \cong \mathbb{C}[G]$ for a torus invariant $G$-constellation $\mathcal{F}$.

[1]This component is also called the coherent component.

For a $G$-prebrick $\Gamma$, by King [12], we have an affine scheme $D(\Gamma)$ parametrising $G$-constellations whose basis is $\Gamma$. The affine scheme $D(\Gamma)$ is not necessarily irreducible, but $D(\Gamma)$ has a distinguished component $U(\Gamma)$ containing the torus $T = (\mathbb{C}^\times)^n/G$. In addition, we can show that $U(\Gamma) = \operatorname{Spec}\mathbb{C}[S(\Gamma)]$ for a semigroup $S(\Gamma)$. If the toric affine variety $U(\Gamma)$ has a torus fixed point, then $\Gamma$ is called a *$G$-brick*. We can prove that $Y_\theta$ is covered by $U(\Gamma)$'s for suitable $G$-bricks $\Gamma$.

On the other hand, from [14, 17], we know that a 3-fold quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if the group $G$ is of type $\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$, i.e.

$$G = \langle \operatorname{diag}(\epsilon, \epsilon^a, \epsilon^{r-a}) \bigm| \epsilon^r = 1\rangle.$$

In this case, the quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein. While $X$ does not admit a crepant resolution, $X$ has the *economic resolution* $\phi\colon Y \to X$ obtained by a toric method called *weighted blowups* (or *Kawamata blowups*). For each step of the weighted blowups, we define three *round down functions*, which are maps between monomial lattices.

As $Y$ is toric, $Y$ is determined by its associated toric fan $\Sigma$ with the lattice $M$ of $G$-invariant monomials. From toric geometry, note that $Y$ is covered by torus invariant affine open subsets $U_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ for $\sigma \in \Sigma_{\max}$ where $\Sigma_{\max}$ denotes the set of maximal cones in $\Sigma$.

Using the round down functions, we find a set $\mathfrak{S}$ of $G$-bricks such that there exists a bijective map $\Sigma_{\max} \to \mathfrak{S}$ sending $\sigma$ to $\Gamma_\sigma$ with $U(\Gamma_\sigma) \cong U_\sigma$. We show that there exists a parameter $\theta \in \Theta$ such that $U(\Gamma_\sigma)$'s cover $Y_\theta$ for $\Gamma_\sigma \in \mathfrak{S}$. This proves that the economic resolution $Y$ is isomorphic to the birational component $Y_\theta$ of $\mathcal{M}_\theta$.

Moreover, we further prove $D(\Gamma) \cong \mathbb{C}^3$ for $\Gamma \in \mathfrak{S}$. So the irreducible component $Y_\theta$ is actually a connected component. We conjecture that the moduli space $\mathcal{M}_\theta$ is irreducible, which implies $Y \cong \mathcal{M}_\theta$.

**Layout of this article.** In Section 2, we define $G$-(pre)bricks and describe the birational component $Y_\theta$ using $G$-bricks. Section 3 explains how to obtain the economic resolutions using toric methods and defines round down functions. In Section 4, we explain how to find $G$-bricks and a parameter $\theta \in \Theta$ such that the economic resolution is isomorphic to the birational component $Y_\theta$. In Section 5 we describe Kedzierski's GIT chamber using the $A_{r-1}$ root system. In Section 6, we calculate $G$-bricks and Kedzierski's GIT chamber for the group of type $\frac{1}{12}(1, 7, 5)$.

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## 2. $G$-bricks and moduli spaces of $G$-constellations

In this section we define a *$G$-prebrick* which is a generalized version of Nakamura's $G$-graph from [16]. By using $G$-prebricks, we describe local charts of moduli spaces of $G$-constellations.

In this section, we restrict ourselves to the case where $G$ is a finite cyclic subgroup of $\mathrm{GL}_3(\mathbb{C})$. It is possible to generalize part of the argument to include general finite small abelian groups in $\mathrm{GL}_n(\mathbb{C})$ for any dimension $n$. However we prefer to focus on this case where we can avoid the difficulty of notation.

2.1. **Moduli spaces of $G$-constellations.** In this section, we review the construction of moduli spaces $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations as described in [2, 12].

Define the group $G = \langle \mathrm{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3}) \,\big|\, \epsilon^r = 1 \rangle \subset \mathrm{GL}_3(\mathbb{C})$. We call $G$ *the group of type* $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. We can identify the set of irreducible representations of $G$ with the character group $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ of $G$. Note that the regular representation $\mathbb{C}[G]$ is isomorphic to $\bigoplus_{\rho \in G^\vee} \mathbb{C}\rho$.

**Definition 2.1.** A *$G$-constellation* on $\mathbb{C}^3$ is a $G$-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^3$ with $\mathrm{H}^0(\mathcal{F})$ isomorphic to the regular representation $\mathbb{C}[G]$ of $G$ as a $\mathbb{C}[G]$-module.

The representation ring $R(G)$ of $G$ is $\bigoplus_{\rho \in G^\vee} \mathbb{Z} \cdot \rho$. Define the GIT stability parameter space

$$\Theta = \left\{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\big|\, \theta\left(\mathbb{C}[G]\right) = 0 \right\}.$$

**Definition 2.2.** For a stability parameter $\theta \in \Theta$, we say that:

(i) a $G$-constellation $\mathcal{F}$ is *$\theta$-semistable* if $\theta(\mathcal{G}) \geq 0$ for every proper submodule $\mathcal{G} \subset \mathcal{F}$.
(ii) a $G$-constellation $\mathcal{F}$ is *$\theta$-stable* if $\theta(\mathcal{G}) > 0$ for every nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.
(iii) $\theta$ is *generic* if every $\theta$-semistable object is $\theta$-stable.

It is known [4] that the language of $G$-constellations is the same as the language of the *McKay quiver representations*. Thus by King [12], the moduli spaces of $G$-constellations can be constructed using Geometric Invariant Theory (GIT).

Let $\mathrm{Rep}\, G$ be the affine scheme whose coordinate ring is

$$\mathbb{C}[\mathrm{Rep}\, G] = \mathbb{C}[x_i, y_i, z_i \,\big|\, i \in G^\vee] \big/ I_G$$

where $I_G$ is the ideal generated by the following quadrics:

$$\begin{cases} x_i y_{i+\alpha_1} - y_i x_{i+\alpha_2}, \\ x_i z_{i+\alpha_1} - z_i x_{i+\alpha_3}, \\ y_i z_{i+\alpha_2} - z_i y_{i+\alpha_3}. \end{cases} \tag{2.3}$$

Let $\delta = (1, \dots, 1) \in \mathbb{Z}^r_{\geq 0}$. The group $\mathrm{GL}(\delta) := \prod_{i \in G^\vee} \mathbb{C}^\times = (\mathbb{C}^\times)^r$ acts on $\operatorname{Rep} G$ via change of basis. For a parameter $\theta \in \Theta$, define the GIT quotient with respect to $\theta$

$$\operatorname{Rep} G /\!\!/_\theta \mathrm{GL}(\delta) := \operatorname{Rep}^{ss}_\theta G / \mathrm{GL}(\delta)$$

parametrising closed $\mathrm{GL}(\delta)$-orbits in $\operatorname{Rep}^{ss}_\theta G$ where $\operatorname{Rep}^{ss}_\theta G$ denotes the $\theta$-semistable locus in $\operatorname{Rep} G$.

**Theorem 2.4** (King [12])**.** *Let us define $\mathcal{M}_\theta := \operatorname{Rep} G /\!\!/_\theta \mathrm{GL}(\delta)$.*

(i) *The quasiprojective scheme $\mathcal{M}_\theta$ is a coarse moduli space of $\theta$-semistable $G$-constellations up to $S$-equivalence.*
(ii) *If $\theta$ is generic, the scheme $\mathcal{M}_\theta$ is a fine moduli space of $\theta$-stable $G$-constellations.*
(iii) *The scheme $\mathcal{M}_\theta$ is projective over $\mathcal{M}_0 = \operatorname{Spec} \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$.*

**Birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$.** Let $\mathcal{M}_\theta$ denote the moduli space of $\theta$-semistable $G$-constellations. Note that the moduli space $\mathcal{M}_\theta$ need not be irreducible [4].

Note that for every parameter $\theta$, there exists a natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$. Indeed, for a $G$-orbit $Z$ in the algebraic torus $\mathbf{T} := (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$, since $Z$ is a free $G$-orbit, $\mathcal{O}_Z$ has no nonzero proper submodules. Thus $\mathcal{O}_Z$ is a $\theta$-stable $G$-constellation. Hence it follows that the torus $T := (\mathbb{C}^\times)^3/G$ is the fine moduli space of $\theta$-stable $G$-constellations supported on $\mathbf{T}$ because any $G$-constellation supporting on a free $G$-orbit $Z$ is isomorphic to $\mathcal{O}_Z$.

**Theorem 2.5** (Craw, Maclagan and Thomas [3])**.** *Let $\theta \in \Theta$ be generic. Then $\mathcal{M}_\theta$ has a unique irreducible component $Y_\theta$ that contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover $Y_\theta$ satisfies the following properties:*

$$\begin{array}{ccc} Y_\theta & \hookrightarrow & \mathcal{M}_\theta \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^3/G & \hookrightarrow & \mathcal{M}_0 \end{array}$$

(i) *$Y_\theta$ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^3/G$.*
(ii) *$Y_\theta$ is projective over the quotient variety $\mathbb{C}^3/G$.*

**Definition 2.6.** The unique irreducible component $Y_\theta$ in Theorem 2.5 is called the *birational component* of $\mathcal{M}_\theta$.

Since Craw, Maclagan and Thomas [3] constructed $Y_\theta$ as GIT quotient of a reduced affine scheme, it follows that $Y_\theta$ is reduced.

**Remark 2.7.** Since $\mathbf{T} = (\mathbb{C}^\times)^3$ acts on $\mathbb{C}^3$, the algebraic torus $\mathbf{T}$ acts on the moduli space $\mathcal{M}_\theta$ naturally. Fixed points of the $\mathbf{T}$-action play a crucial role in the study of the moduli space $\mathcal{M}_\theta$. ♢

2.2. **$G$-prebricks and local charts of $\mathcal{M}_\theta$.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3),$$

which is an overlattice of $\overline{L} = \mathbb{Z}^3$ of finite index. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$. Set $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The two dual lattices $\overline{M}$ and $M$ can be identified with Laurent monomials and $G$-invariant Laurent monomials, respectively. The embedding of $G$ into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee$$

whose kernel is $M$. Note that there are two isomorphisms of abelian groups $L/\mathbb{Z}^3 \to G$ and $\overline{M}/M \to G^\vee$.

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.

$$\overline{M}_{\geq 0} = \{x^{m_1} y^{m_2} z^{m_3} \in \overline{M} \mid m_1, m_2, m_3 \geq 0\}.$$

For a set $A \subset \mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$, let $\langle A \rangle$ denote the $\mathbb{C}[x, y, z]$-submodule of $\mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ generated by $A$.

Let $\sigma_+$ be the cone in $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ generated by $e_1, e_2, e_3$. Note that the corresponding affine toric variety $U_{\sigma_+} = \mathrm{Spec}\, \mathbb{C}[\sigma_+^\vee \cap M]$ is isomorphic to the quotient variety $\mathbb{C}^3/G = \mathrm{Spec}\, \mathbb{C}[x, y, z]^G$.

**Definition 2.8.** A *$G$-prebrick* $\Gamma$ is a subset of Laurent monomials in $\mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ satisfying:

(i) the monomial $\mathbf{1}$ is in $\Gamma$.
(ii) for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ of weight $\rho$, i.e. $\mathrm{wt}\colon \Gamma \to G^\vee$ is bijective.
(iii) if $\mathbf{n}' \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{n}, \mathbf{n}' \in \overline{M}_{\geq 0}$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.
(iv) the set $\Gamma$ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path in $\Gamma$ from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of $x, y, z$.

For a Laurent monomial $\mathbf{m} \in \overline{M}$, let $\mathrm{wt}_\Gamma(\mathbf{m})$ denote the unique element $\mathbf{m}_\rho$ in $\Gamma$ of the same weight as $\mathbf{m}$.

**Remark 2.9.** Nakamura's $G$-graph $\Gamma$ in [16] is a $G$-prebrick because if a monomial $\mathbf{n}' \cdot \mathbf{n}$ is in $\Gamma$ for two monomials $\mathbf{n}, \mathbf{n}' \in \overline{M}_{\geq 0}$, then $\mathbf{n}$ is in $\Gamma$. The main difference between $G$-graphs and $G$-prebricks is that elements of $G$-prebricks are allowed to be Laurent monomials, not just genuine monomials. $\diamondsuit$

**Example 2.10.** Let $G$ be the group of type $\frac{1}{7}(1, 3, 4)$. Then

$$\Gamma_1 = \left\{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\},$$

$$\Gamma_2 = \left\{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z}, \tfrac{y^3}{z^2}\right\}$$

are $G$-prebricks. For $\Gamma_1$, we have $\mathrm{wt}_{\Gamma_1}(x) = \frac{z}{y}$ and $\mathrm{wt}_{\Gamma_1}(y^3) = \frac{z^2}{y^2}$. $\Diamond$

For a $G$-prebrick $\Gamma = \{\mathbf{m}_\rho\}$, as an analogue of [16], define $S(\Gamma)$ to be the subsemigroup of $M$ generated by $\dfrac{\mathbf{n}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho)}$ for all $\mathbf{n} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. Define a cone $\sigma(\Gamma)$ in $L_\mathbb{R} = \mathbb{R}^3$ as follows:

$$\begin{aligned}\sigma(\Gamma) &= S(\Gamma)^\vee \\ &= \left\{\mathbf{u} \in L_\mathbb{R} \,\middle|\, \left\langle \mathbf{u}, \frac{\mathbf{n}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho)}\right\rangle \geq 0, \quad \forall \mathbf{m}_\rho \in \Gamma,\ \mathbf{n} \in \overline{M}_{\geq 0}\right\}.\end{aligned}$$

Observe that:

(i) $\left(\overline{M}_{\geq 0} \cap M\right) \subset S(\Gamma)$,
(ii) $\sigma(\Gamma) \subset \sigma_+$,
(iii) $S(\Gamma) \subset \left(\sigma(\Gamma)^\vee \cap M\right)$.

**Lemma 2.11.** *Let $\Gamma$ be a $G$-prebrick. Define*

$$B(\Gamma) := \left\{\mathbf{f}\cdot\mathbf{m}_\rho \,\middle|\, \mathbf{m}_\rho \in \Gamma,\ \mathbf{f} \in \{x, y, z\}\right\} \backslash \Gamma.$$

*Then the semigroup $S(\Gamma)$ is generated by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$ as a semigroup. In particular, $S(\Gamma)$ is finitely generated as a semigroup.*

*Proof.* Let $S$ be the subsemigroup of $M$ generated by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$. Clearly, $S \subset S(\Gamma)$. For the opposite inclusion, it is enough to show that the generators of $S(\Gamma)$ are in $S$.

An arbitrary generator of $S(\Gamma)$ is of the form $\frac{\mathbf{n}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho)}$ for some $\mathbf{n} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. We may assume that $\mathbf{n}\cdot\mathbf{m}_\rho \notin \Gamma$. In particular, $\mathbf{n} \neq \mathbf{1}$. Since $\mathbf{n}$ has positive degree, there exists $\mathbf{f} \in \{x, y, z\}$ such that $\mathbf{f}$ divides $\mathbf{n}$, i.e. $\frac{\mathbf{n}}{\mathbf{f}} \in \overline{M}_{\geq 0}$ and $\deg(\frac{\mathbf{n}}{\mathbf{f}}) < \deg(\mathbf{n})$. Let $\mathbf{m}_{\rho'}$ denote $\mathrm{wt}_\Gamma(\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho)$. Note that

$$\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'}) = \mathrm{wt}_\Gamma(\mathbf{f}\cdot\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho) = \mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho).$$

Thus

$$\begin{aligned}\frac{\mathbf{n}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho)} &= \frac{\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho)} \cdot \frac{\mathbf{f}\cdot\mathrm{wt}_\Gamma(\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho)}{\mathrm{wt}_\Gamma(\mathbf{n}\cdot\mathbf{m}_\rho)} \\ &= \frac{\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{n}}{\mathbf{f}}\cdot\mathbf{m}_\rho)} \cdot \frac{\mathbf{f}\cdot\mathbf{m}_{\rho'}}{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'})}.\end{aligned}$$

By induction on the degree of monomial $\mathbf{n}$, the assertion is proved. $\square$

The set $B(\Gamma)$ in the lemma above is called the *Border bases* of $\Gamma$. As $B(\Gamma)$ is finite, the semigroup $S(\Gamma)$ is finitely generated as a semigroup. Thus the semigroup $S(\Gamma)$ defines an affine toric variety. Define two affine toric varieties:

$$\begin{aligned}U(\Gamma) &:= \operatorname{Spec}\mathbb{C}[S(\Gamma)], \\ U^\nu(\Gamma) &:= \operatorname{Spec}\mathbb{C}[\sigma(\Gamma)^\vee \cap M].\end{aligned}$$

Note that the torus $\operatorname{Spec}\mathbb{C}[M]$ of $U(\Gamma)$ is isomorphic to $T = (\mathbb{C}^\times)^3/G$ and that $U^\nu(\Gamma)$ is the normalization of $U(\Gamma)$.

Craw, Maclagan and Thomas [4] showed that there exists a torus invariant $G$-cluster which does not lie in the birational component $Y_\theta$. The following definition is implicit in [4].

**Definition 2.12.** A $G$-prebrick $\Gamma$ is called a *$G$-brick* if the affine toric variety $U(\Gamma)$ contains a torus fixed point.

From toric geometry, $U(\Gamma)$ has a torus fixed point if and only if $S(\Gamma) \cap (S(\Gamma))^{-1} = \{\mathbf{1}\}$, i.e. the cone $\sigma(\Gamma)$ is a 3-dimensional cone.

**Example 2.13.** Consider the $G$-prebricks $\Gamma_1, \Gamma_2$ in Example 2.10. By Lemma 2.11, $S(\Gamma_1)$ is generated by $\frac{y^5}{z^2}, \frac{z^3}{y^4}, \frac{xy}{z}$. We have

$$\begin{aligned}\sigma(\Gamma_1) &= \left\{\mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m}\rangle \geq 0, \text{ for all } \mathbf{m} \in \left\{\tfrac{y^5}{z^2}, \tfrac{z^3}{y^4}, \tfrac{xy}{z}\right\}\right\}, \\ &= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(3,2,5), \tfrac{1}{7}(1,3,4)\right).\end{aligned}$$

Similarly, we can see that

$$\begin{aligned}\sigma(\Gamma_2) &= \left\{\mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m}\rangle \geq 0, \text{ for all } \mathbf{m} \in \left\{\tfrac{y^4}{z^3}, \tfrac{z^4}{y^3}, \tfrac{xz^2}{y^3}\right\}\right\}, \\ &= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(6,4,3)\right).\end{aligned}$$

Since $S(\Gamma_1) = \sigma(\Gamma_1)^\vee \cap M$ and $S(\Gamma_2) = \sigma(\Gamma_2)^\vee \cap M$, the two $G$-prebricks $\Gamma_1$, $\Gamma_2$ are $G$-bricks. Moreover the two toric varieties $U(\Gamma_1)$ and $U(\Gamma_2)$ are smooth. ♢

Let $\Gamma$ be a $G$-prebrick. Define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle.$$

The module $C(\Gamma)$ is a torus invariant $G$-constellation. A submodule $\mathcal{G}$ of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a $\mathbb{C}$-basis of $\mathcal{G}$.

**Lemma 2.14.** *Let $A$ be a subset of $\Gamma$. The following are equivalent.*

(i) *The set $A$ forms a $\mathbb{C}$-basis of a submodule of $C(\Gamma)$.*
(ii) *If $\mathbf{m}_\rho \in A$ and $\mathbf{f} \in \{x, y, z\}$, then $\mathbf{f} \cdot \mathbf{m}_\rho \in \Gamma$ implies $\mathbf{f} \cdot \mathbf{m}_\rho \in A$.*

Let $p$ be a point in $U(\Gamma)$. Then the evaluation map

$$\mathrm{ev}_p \colon S(\Gamma) \to (\mathbb{C}, \times),$$

is a semigroup homomorphism.

To assign a $G$-constellation $C(\Gamma)_p$ to the point $p$ of $U(\Gamma)$, first consider the $\mathbb{C}$-vector space with basis $\Gamma$ whose $G$-action is induced by the $G$-action on $\mathbb{C}[x, y, z]$. Endow it with the following $\mathbb{C}[x, y, z]$-action:

$$\mathbf{n} * \mathbf{m}_\rho := \mathrm{ev}_p\left(\frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)}\right) \mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho), \tag{2.15}$$

for a monomial $\mathbf{n} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$.

**Lemma 2.16.** *Let $\Gamma$ be a $G$-prebrick.*

(i) *For every $p \in U(\Gamma)$, $C(\Gamma)_p$ is a $G$-constellation.*
(ii) *For every $p \in U(\Gamma)$, $\Gamma$ is a $\mathbb{C}$-basis of $C(\Gamma)_p$.*
(iii) *If $p$ and $q$ are different points in $U(\Gamma)$, then $C(\Gamma)_p \ncong C(\Gamma)_q$.*
(iv) *Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free $G$-orbit and $p$ the corresponding point in the torus $\operatorname{Spec}\mathbb{C}[M]$ of $U(\Gamma)$. Then $C(\Gamma)_p \cong \mathcal{O}_Z$ as $G$-constellations.*
(v) *If $U(\Gamma)$ has a torus fixed point $p$, then $C(\Gamma)_p \cong C(\Gamma)$.*

*Proof.* From the definition of $C(\Gamma)_p$, the assertions (i), (ii) and (v) follow immediately. The assertion (iii) follows from the fact that points on the affine toric variety $U(\Gamma)$ are in 1-to-1 correspondence with semigroup homomorphisms from $S(\Gamma)$ to $\mathbb{C}$.

It remains to show (iv). Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free $G$-orbit and $p$ the corresponding point in $\operatorname{Spec}\mathbb{C}[M] \subset U(\Gamma)$. There is a surjective $G$-equivariant $\mathbb{C}[x,y,z]$-module homomorphism

$$\mathbb{C}[x,y,z] \to C(\Gamma)_p \quad \text{given by } f \mapsto f * \mathbf{1}.$$

whose kernel is equal to the ideal of $Z$. This proves (iv). ☐

**Definition 2.17.** A $G$-prebrick is said to be *$\theta$-stable* if $C(\Gamma)$ is $\theta$-stable.

**Deformation space $D(\Gamma)$.** We introduce deformation theory of $C(\Gamma)$ for a $\theta$-stable $G$-prebrick $\Gamma$. We deform $C(\Gamma)$, keeping the same vector space structure, but perturbing the structure of $\mathbb{C}[x,y,z]$-module. Since we fix a $\mathbb{C}$-basis $\Gamma$ of $C(\Gamma)$, deforming $C(\Gamma)$ involves $3r$ parameters $\{x_\rho, y_\rho, z_\rho \,|\, \rho \in G^\vee\}$ with

$$\begin{cases} x * \mathbf{m}_\rho = x_\rho \operatorname{wt}_\Gamma(x \cdot \mathbf{m}_\rho), \\ y * \mathbf{m}_\rho = y_\rho \operatorname{wt}_\Gamma(y \cdot \mathbf{m}_\rho), \\ z * \mathbf{m}_\rho = z_\rho \operatorname{wt}_\Gamma(z \cdot \mathbf{m}_\rho), \end{cases}$$

with the following commutation relations:

$$\begin{cases} x_\rho y_{\operatorname{wt}(x\cdot\mathbf{m}_\rho)} - y_\rho x_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)}, \\ x_\rho z_{\operatorname{wt}(x\cdot\mathbf{m}_\rho)} - z_\rho x_{\operatorname{wt}(z\cdot\mathbf{m}_\rho)}, \\ y_\rho z_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)} - z_\rho y_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)}. \end{cases} \tag{2.18}$$

Note that $\operatorname{wt}_\Gamma(\mathbf{m}) \in \Gamma$ is the base of the same weight as $\mathbf{m}$. Fixing a basis $\Gamma$ means that we set $\mathbf{f}_\rho = 1$ if $\operatorname{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho) = \mathbf{f}\cdot\mathbf{m}_\rho$ for $\mathbf{f} \in \{x,y,z\}$. Define a subset of the $3r$ parameters

$$\Lambda(\Gamma) := \big\{\mathbf{f}_\rho \,\big|\, \operatorname{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho) = \mathbf{f}\cdot\mathbf{m}_\rho,\ \mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}\big\},$$

i.e. $\Lambda(\Gamma)$ is the set of parameters fixed to be 1. Define the affine scheme

$$D(\Gamma) := \operatorname{Spec}\big(\mathbb{C}[x_\rho, y_\rho, z_\rho \,|\, \rho \in G^\vee]/I_\Gamma\big) \tag{2.19}$$

where $I_\Gamma = \big\langle \text{the quadrics in (2.18)}, \mathbf{f}_\rho - 1 \,\big|\, \mathbf{f}_\rho \in \Lambda(\Gamma)\big\rangle$. By King [12], the affine scheme $D(\Gamma)$ is an open set of $\mathcal{M}_\theta$ containing the point

corresponding to $C(\Gamma)$. More precisely, consider an affine open set $\widetilde{U_\Gamma}$ in $\operatorname{Rep} G$, which is defined by $\mathbf{f}_\rho$ to be nonzero for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$. Note that $\widetilde{U_\Gamma}$ is $\mathrm{GL}(\delta)$-invariant and that $\widetilde{U_\Gamma}$ is in the $\theta$-stable locus. Since the quotient map $\operatorname{Rep}^{ss}_\theta G \to \mathcal{M}_\theta$ is a geometric quotient for generic $\theta$, from GIT [15], it follows that $\operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$ is an affine open subset of $\mathcal{M}_\theta$. On the other hand, setting $\mathbf{f}_\rho \in \Lambda(\Gamma)$ to be 1 for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$ gives a slice of the $\mathrm{GL}(\delta)$-action. Thus $D(\Gamma)$ is isomorphic to $\operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$.

**Remark 2.20.** The affine open subset $D(\Gamma)$ of the moduli space $\mathcal{M}_\theta$ parametrises $G$-constellations whose basis is $\Gamma$. ♢

**Proposition 2.21.** *For generic $\theta$, let $\Gamma$ be a $\theta$-stable $G$-brick and $Y_\theta$ the birational component of $\mathcal{M}_\theta$. Then $C(\Gamma)_p$ is $\theta$-stable for every $p \in U(\Gamma)$. Furthermore, there exists an open immersion*

$$U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)] \ \hookrightarrow \ Y_\theta \subset \mathcal{M}_\theta,$$

*which fits into the following commutative diagram:*

$$\begin{array}{ccc} U(\Gamma) & \hookrightarrow & Y_\theta \\ \downarrow & & \downarrow \\ D(\Gamma) & \hookrightarrow & \mathcal{M}_\theta \end{array}$$

*where the vertical morphisms are closed embeddings.*

*Proof.* Let us assume that the $G$-constellation $C(\Gamma)$ is $\theta$-stable. Let $p$ be an arbitrary point in $U(\Gamma)$ and $\mathcal{G}$ a submodule of $C(\Gamma)_p$. By the definition of $C(\Gamma)_p$, there is a submodule $\mathcal{G}'$ of $C(\Gamma)$ whose support is the same as $\mathcal{G}$. Since $C(\Gamma)$ is $\theta$-stable, $\theta(\mathcal{G}) = \theta(\mathcal{G}') > 0$. Thus $C(\Gamma)_p$ is $\theta$-stable.

Since there is a $\mathbb{C}$-algebra epimorphism from $\mathbb{C}[D(\Gamma)]$ to $\mathbb{C}[S(\Gamma)]$ given by

$$\mathbf{f}_\rho \mapsto \frac{\mathbf{f} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_\rho)}$$

for $\mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}$, it follows that $U(\Gamma)$ is a closed subscheme of $D(\Gamma)$.

As Craw, Maclagan, and Thomas [3] proved that the birational component $Y_\theta$ is a unique irreducible component of $\mathcal{M}_\theta$ containing the torus $T = (\mathbb{C}^\times)^3/G$, it follows that $Y_\theta \cap D(\Gamma)$ is a unique irreducible component of $D(\Gamma)$ containing the torus $T$.

The morphism $U(\Gamma) \hookrightarrow D(\Gamma) \subset \mathcal{M}_\theta$ induces an isomorphism between the torus $\operatorname{Spec} \mathbb{C}[M]$ and the torus $T$ of $Y_\theta$ by Lemma 2.16 (iv). Note that $U(\Gamma)$ is in the component $Y_\theta \cap D(\Gamma)$ because $U(\Gamma)$ is a closed subset of $D(\Gamma)$ containing $T$. Since both $U(\Gamma)$ and $Y_\theta \cap D(\Gamma)$ are reduced and of the same dimension, $U(\Gamma)$ is equal to $Y_\theta \cap D(\Gamma)$. Thus there exists an open immersion from $U(\Gamma)$ to $Y_\theta$. □

2.3. **$G$-bricks and the biratioanl component $Y_\theta$.** In this section, we present a 1-to-1 correspondence between the set of torus fixed points in $Y_\theta$ and the set of $\theta$-stable $G$-bricks.

**Proposition 2.22.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. For a generic parameter $\theta$, there is a 1-to-1 correspondence between the set of torus fixed points in the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ and the set of $\theta$-stable $G$-bricks.*

*Proof.* [2] In Section 2.2, we have seen that if $\Gamma$ is a $\theta$-stable $G$-brick, then $C(\Gamma)$ is a torus invariant $G$-constellation corresponding to a torus fixed point in $Y_\theta$.

Let $p \in Y_\theta$ be a torus fixed point and $\mathcal{F}$ the corresponding torus invariant $G$-constellation. For a one parameter subgroup

$$\lambda \colon \mathbb{C}^\times \to T \subset Y_\theta$$

with $\lim_{t \to 0} \lambda(t) = p$, $\lambda$ induces a flat family $\mathcal{V}$ of $G$-constellations over $\mathbb{A}^1_{\mathbb{C}}$. Note that $\mathcal{V}$ has generic support; for every nonzero $t \in \mathbb{A}^1_{\mathbb{C}}$, the $G$-constellation $\mathcal{V}_t$ over $t$ is isomorphic to $\mathcal{O}_Z$ for a free $G$-orbit $Z$ in $\mathbf{T} = (\mathbb{C}^\times)^3$. There is a set $\Gamma = \{\mathbf{m}_\rho \in \overline{M} \mid \rho \in G^\vee\}$ satisfying:

(1) $\Gamma$ is a $\mathbb{C}$-basis of $\mathcal{V}_t$ for every $t \in \mathbb{A}^1_{\mathbb{C}}$.
(2) $\mathbf{1} \in \Gamma$.
(3) $\mathcal{V}_t$ is isomorphic to $\langle \Gamma \rangle / N_t$ for a submodule $N_t$ of $\langle \Gamma \rangle$, where $\langle \Gamma \rangle$ denotes the $\mathbb{C}[x, y, z]$-module generated by $\Gamma$.

We prove that $\Gamma$ is a $G$-prebrick and that $\mathcal{F} \cong C(\Gamma)$. Note that $\mathcal{F}$ can be written as $\langle \Gamma \rangle / N$ for a submodule $N$. For any $\mathbf{m}_\rho \in \Gamma$, since $\mathbf{m}_\rho$ is a base, $\mathbf{m}_\rho$ is not in $N$. Moreover if $\mathbf{n} \cdot \mathbf{m}_\rho \notin \Gamma$ for $\mathbf{n} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in N$ because the dimension of $\mathrm{H}^0(\mathcal{F})$ is $r = |\Gamma|$. This proves that $N = \langle B(\Gamma) \rangle$, where $B(\Gamma)$ is the Border bases in Lemma 2.11. From this, it follows that $\Gamma$ satisfies the conditions (i),(ii),(iii) in Definition 2.8. As $\mathcal{F}$ is $\theta$-stable for generic $\theta$, the connectedness condition (iv) follows.

To see that the $G$-prebrick $\Gamma$ is a $G$-brick, note that the point $p \in Y_\theta$ corresponds to the isomorphism class of $C(\Gamma)$ so $p \in \mathrm{D}(\Gamma)$. Thus $p$ is in $U(\Gamma) = Y_\theta \cap D(\Gamma)$. $\square$

**Corollary 2.23.** *Let $\Gamma$ be a $G$-prebrick. Then $C(\Gamma)$ lies in the birational component $Y_\theta$ if and only if $\Gamma$ is a $G$-brick.*

**Theorem 2.24.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be a finite diagonal group and $\theta$ a generic GIT parameter for $G$-constellations. Assume that $\mathfrak{S}$ is the set of all $\theta$-stable $G$-bricks.*

(i) *The birational component $Y_\theta$ of $\mathcal{M}_\theta$ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$.*

[2] In [8], there is another proof using the language of the McKay quiver representations.

(ii) *The normalization of $Y_\theta$ is isomorphic to the normal toric variety whose toric fan consists of the 3-dimensional cones $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{S}$ and their faces.*

*Proof.* Let $G$ be the group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. Consider the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$.

Let $Y_\theta$ be the birational component of the moduli space of $\theta$-stable $G$-constellations and $Y_\theta^\nu$ the normalization of $Y_\theta$. Let $Y$ denote the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$. Define the fan $\Sigma$ in $L_\mathbb{R}$ whose maximal cones are $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{S}$. Note that the corresponding toric variety $Y^\nu := X_\Sigma$ is the normalization of $Y$.

By Proposition 2.21, there is an open immersion $\psi\colon Y \to Y_\theta$. From Proposition 2.22, it follows that the image $\psi(Y)$ contains all torus fixed points of $Y_\theta$. The induced morphism $\psi^\nu\colon Y^\nu \to Y_\theta^\nu$ is an open embedding of normal toric varieties with the same number of torus fixed points. Thus the morphism $\psi^\nu$ should be an isomorphism. This proves (ii).

To show (i), suppose that $Y_\theta \setminus \psi(Y)$ is nonempty so it contains a torus orbit $O$ of dimension $d \geq 1$. Since the normalization morphism is torus equivariant and surjective, there exists a torus orbit $O'$ in $Y^\nu \cong Y_\theta^\nu$ of dimension $d$ which is mapped to the torus orbit $O$. At the same time, from the fact that $Y^\nu$ is the normalization of $Y$ and that the normalization morphism is finite, it follows that the image of $O'$ is a torus orbit of dimension $d$, so the image is $O$. Thus $O$ is in $\psi(Y)$, which is a contradiction. □

**Corollary 2.25.** *With the notation as in Theorem 2.24, $Y_\theta$ is a normal toric variety if and only if $S(\Gamma) = \sigma(\Gamma)^\vee \cap M$ for all $\Gamma \in \mathfrak{S}$.*

## 3. Weighted blowups and economic resolutions

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$, i.e.

$$G = \langle \mathrm{diag}(\epsilon, \epsilon^a, \epsilon^{r-a}) \,\big|\, \epsilon^r = 1 \rangle.$$

The quotient $X = \mathbb{C}^3/G$ has terminal singularities and has no crepant resolution. However there exist a special kind of toric resolutions, which can be obtained by a sequence of weighted blowups. In this section, we review the notion of toric weighted blowups and define round down functions which are used for finding admissible $G$-bricks.

### 3.1. Terminal quotient singularities in dimension 3.

In this section, we collect various facts from birational geometry. Most of these are taken from [17].

**Definition 3.1.** Let $X$ be a normal quasiprojective variety and $K_X$ the canonical divisor on $X$. We say that $X$ has *terminal singularities* (resp. *canonical singularities*) if it satisfies the following conditions:

(i) there is a positive integer $r$ such that $rK_X$ is a Cartier divisor.
(ii) if $\varphi\colon Y \to X$ is a resolution with $E_i$ prime exceptional divisors such that

$$rK_Y = \varphi^*(rK_X) + r\sum a_i E_i,$$

then $a_i > 0$ (resp. $\geq 0$) for all $i$.

In the definition above, $a_i$ is called the *discrepancy* of $E_i$. A *crepant resolution* $\varphi$ of $X$ is a resolution with all discrepancies zero.

**Birational geometry of toric varieties.** Let $L$ be a lattice of rank $n$ and $M$ the dual lattice of $L$. Let $\sigma$ be a cone in $L \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a primitive element $v \in L \cap \sigma$. The *barycentric subdivision* of $\sigma$ at $v$ is the minimal fan containing all cones $\mathrm{Cone}(\tau, v)$ where $\tau$ varies over all subcones of $\sigma$ with $v \notin \tau$.

**Proposition 3.2** (see e.g. [17])**.** *Let $\Sigma$ be the barycentric subdivision of an $n$-dimensional cone $\sigma$ at $v$. Let $X := U_\sigma$ be the affine toric variety corresponding to $\sigma$ and $Y$ the toric variety corresponding to the fan $\Sigma$.*

(i) *The barycentric subdivision induces a projective toric morphism*

$$\varphi\colon Y \to X.$$

(ii) *The set of 1-dimensional cones of $\Sigma$ consists of 1-dimensional cones of $\sigma$ and* $\mathrm{Cone}(v)$.
(iii) *The torus invariant prime divisor $D_v$ corresponding to the 1-dimensional cone* $\mathrm{Cone}(v)$ *is a $\mathbb{Q}$-Cartier divisor on $Y$.*

*Furthermore if $v$ is an interior lattice point in $\sigma$, then*

$$K_Y = \varphi^*(K_X) + (\langle x_1x_2\cdots x_n, v\rangle - 1)D_v,$$

*i.e. the discrepancy of the exceptional divisor $D_v$ is $\langle x_1x_2\dots x_n, v\rangle - 1$.*

**Example 3.3.** Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z}\cdot\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ the dual lattice. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and $\sigma_+$ the cone generated by $e_1, e_2, e_3$. As in Section 2.2, the toric variety $X := U_{\sigma_+}$ is the quotient variety $\mathbb{C}^3/G$ where $G$ is the group of type $\frac{1}{r}(1, a, r-a)$.

Set $v_i := \frac{1}{r}(i, \overline{ai}, \overline{r-ai}) \in L$ for each $1 \leq i < r-1$ where $^{-}$ denotes the residue modulo $r$. Let $E_i$ be the torus invariant prime divisor corresponding to $v_i$. From Proposition 3.2, the discrepancy of $E_i$ is

$$\frac{i}{r} + \frac{\overline{ai}}{r} + \frac{r-\overline{ai}}{r} - 1 = \frac{i}{r} > 0.$$

This shows that $X$ has only terminal singularities. $\diamondsuit$

We have seen that the quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if $G$ is the group of type $\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$. Conversely, these groups are essentially all the cases, by the following.

**Theorem 3.4** (Morrison and Stevens [14]). *A 3-fold cyclic quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if the group $G \subset \mathrm{GL}_3(\mathbb{C})$ is of type $\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$.*

3.2. **Weighted blowups and round down functions.** Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r-a)$. Set $\overline{L} = \mathbb{Z}^3 \subset L$. Consider the two dual lattices $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$. Note that a (Laurent) monomial $\mathbf{m} \in \overline{M}$ is $G$-invariant if and only if $\mathbf{m}$ is in $M$. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and $\sigma_+$ the cone generated by $e_1, e_2, e_3$. Then $\operatorname{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$ is the quotient variety $X = \mathbb{C}^3/G$. Set $v = \frac{1}{r}(1, a, r-a) \in L$, which corresponds to the exceptional divisor of the smallest discrepancy (see Example 3.3). Define three cones

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v).$$

Define $\Sigma$ to be the fan consisting of the three cones $\sigma_1, \sigma_2, \sigma_3$ and their faces. The fan $\Sigma$ is the barycentric subdivision of $\sigma_+$ at $v$. Let $Y_1$ be the toric variety corresponding to the fan $\Sigma$ together with the lattice $L$. The induced toric morphism $\varphi \colon Y_1 \to X$ is called *the weighted blowup* of $X$ with weight $(1, a, r-a)$.

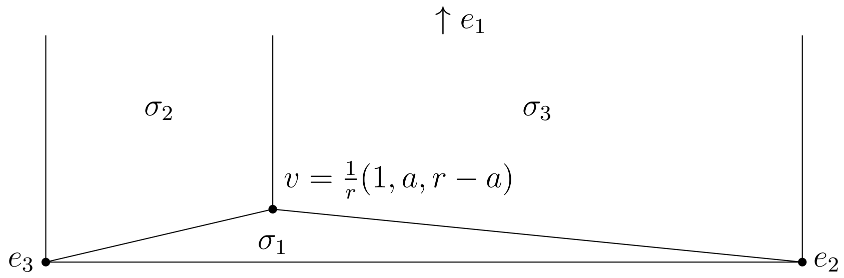


FIGURE 3.1. Weighted blowup of weight $(1, a, r-a)$

Let us consider the sublattice $L_2$ of $L$ generated by $e_1, v, e_3$. Define $M_2 := \mathrm{Hom}_{\mathbb{Z}}(L_2, \mathbb{Z})$ with the dual basis

$$\xi_2 := xy^{-\frac{1}{a}}, \quad \eta_2 := y^{\frac{r}{a}}, \quad \zeta_2 := y^{\frac{a-r}{a}} z.$$

The lattice inclusion $L_2 \hookrightarrow L$ induces a toric morphism

$$\varphi \colon \operatorname{Spec} \mathbb{C}[\sigma_2^\vee \cap M_2] \to U_2 := \operatorname{Spec} \mathbb{C}[\sigma_2^\vee \cap M].$$

Since $\mathbb{C}[\sigma_2^\vee \cap M_2] \cong \mathbb{C}[\xi_2, \eta_2, \zeta_2]$ and the group $G_2 := L/L_2$ is of type $\frac{1}{a}(1, -r, r)$ with eigencoordinates $\xi_2, \eta_2, \zeta_2$, the open subset $U_2$ has a quotient singularity of type $\frac{1}{a}(1, -r, r)$.

Similarly, consider the sublattice $L_3$ of $L$ generated by $e_1, e_2, v$. Let us define the lattice $M_3 := \mathrm{Hom}_{\mathbb{Z}}(L_3, \mathbb{Z})$ with basis

$$\xi_3 := xz^{-\frac{1}{r-a}}, \quad \eta_3 := yz^{\frac{-a}{r-a}}, \quad \zeta_3 := z^{\frac{r}{r-a}}.$$

The open set $U_3 = \operatorname{Spec} \mathbb{C}[\xi_3, \eta_3, \zeta_3]$ has a quotient singularity of type $\frac{1}{r-a}(1, r, -r)$ with eigencoordinates $\xi_3, \eta_3, \zeta_3$. Set $G_3 := L/L_3$.

Lastly, consider the sublattice $L_1$ of $L$ generated by $v, e_2, e_3$. Let us define $M_1 := \mathrm{Hom}_{\mathbb{Z}}(L_1, \mathbb{Z})$ with the dual basis

$$\xi_1 := x^{\frac{1}{r}}, \quad \eta_1 := x^{-\frac{a}{r}} y, \quad \zeta_1 := x^{-\frac{r-a}{r}} z.$$

Since $\{v, e_2, e_3\}$ forms a $\mathbb{Z}$-basis of $L$, i.e. $G_1 = L/L_1$ is the trivial group, the open set $U_1 = \operatorname{Spec} \mathbb{C}[\xi_1, \eta_1, \zeta_1]$ is smooth.

**Example 3.5.** Let $G$ be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.10. The toric fan of the weighted blowup with weight $(1,3,4)$ is shown in Figure 3.2.

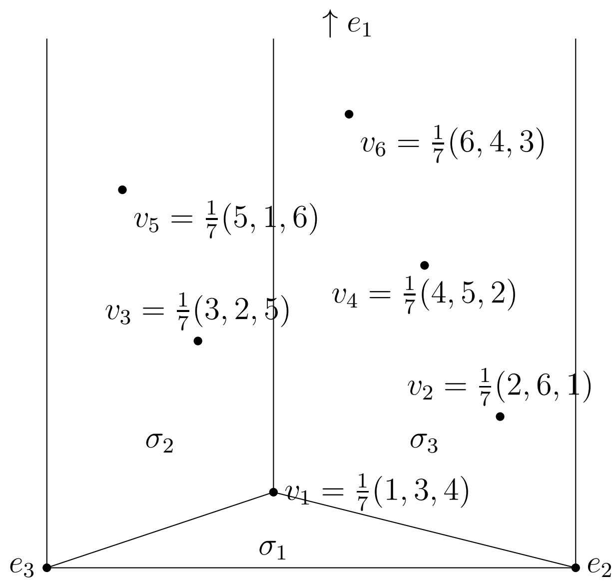


FIGURE 3.2. Weighted blowup of weight $(1,3,4)$

The affine toric variety corresponding to the cone $\sigma_2$ on the left side of $v = \frac{1}{7}(1,3,4)$ has a quotient singularity of type $\frac{1}{3}(1,2,1)$ with eigencoordinates $xy^{-\frac{1}{3}}, y^{\frac{7}{3}}, y^{-\frac{4}{3}}z$. The affine toric variety corresponding to the cone $\sigma_3$ on the right side of $v$ has a singularity of type $\frac{1}{4}(1,3,1)$ with eigencoordinates $xz^{-\frac{1}{4}}, yz^{-\frac{3}{4}}, z^{\frac{7}{4}}$. On the other hand, the affine toric variety corresponding to the cone $\sigma_1 = \mathrm{Cone}(e_2, e_3, v)$ is smooth as $e_2, e_3, v$ form a $\mathbb{Z}$-basis of $L$. $\diamondsuit$

**Definition 3.6** (Round down functions)**.** With the notation above, the *left round down function* $\phi_2 \colon \overline{M} \to M_2$ of the weighted blowup with weight $(1, a, r-a)$ is defined by

$$\phi_2(x^{m_1} y^{m_2} z^{m_3}) = \xi_2^{m_1} \eta_2^{\lfloor \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \rfloor} \zeta_2^{m_3}.$$

where $\lfloor \ \rfloor$ is the floor function. In a similar manner, the *right round down function* $\phi_3 \colon \overline{M} \to M_3$ of the weighted blowup with weight

$(1,a,r-a)$ is defined by

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \xi_3^{m_1}\eta_3^{m_2}\zeta_3^{\lfloor \frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3\rfloor},$$

and the *central round down function* $\phi_1\colon \overline{M} \to M_1$ of the weighted blowup with weight $(1,a,r-a)$ by

$$\phi_1(x^{m_1}y^{m_2}z^{m_3}) = \xi_1^{\lfloor \frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3\rfloor}\eta_1^{m_2}\zeta_1^{m_3}.$$

**Lemma 3.7.** *For each $k=1,2,3$, let $\phi_k$ be the round down function of the weighted blowup with weight $(1,a,r-a)$. For a monomial $\mathbf{m}\in\overline{M}$ and a $G$-invariant monomial $\mathbf{n}\in M$,*

$$\phi_k(\mathbf{m}\cdot\mathbf{n}) = \phi_k(\mathbf{m})\cdot\mathbf{n}.$$

*Thus the weight of $\phi_k(\mathbf{m}\cdot\mathbf{n})$ and the weight of $\phi_k(\mathbf{m})$ are the same in terms of the $G_k$-action. Therefore $\phi_k$ induces a surjective map*

$$\phi_k\colon G^\vee \to G_k^\vee, \quad \rho\mapsto\phi_k(\rho),$$

*where $\phi_k(\rho)$ is the weight of $\phi_k(\mathbf{m})$ for a monomial $\mathbf{m}\in\overline{M}$ of weight $\rho$.*

*Proof.* Since $M_k$ contains $M$ as the lattice of $G_k$-invariant monomials, $\mathbf{n}$ is in $M_k$. By definition, the assertions follow. □

**Remark 3.8.** Davis, Logvinenko, and Reid [6] introduced a related construction in a more general setting. ◊

**Lemma 3.9.** *For each $k=1,2,3$, let $\phi_k$ be the round down function of the weighted blowup with weight $(1,a,r-a)$. Let $\mathbf{m}\in\overline{M}$ be a Laurent monomial of weight $j$.*

(i) *If $0\le j<r-a$, then $\phi_2(y\cdot\mathbf{m})=\phi_2(\mathbf{m})$.*
(ii) *If $0\le j<a$, then $\phi_3(z\cdot\mathbf{m})=\phi_3(\mathbf{m})$.*
(iii) *If $0\le j<r-1$, then $\phi_1(x\cdot\mathbf{m})=\phi_1(\mathbf{m})$.*
(iv) *If $\phi_k(\mathbf{m})=\phi_k(\mathbf{m}')$, then $\mathbf{m}=\mathbf{n}\cdot\mathbf{m}'$ or $\mathbf{m}'=\mathbf{n}\cdot\mathbf{m}$ for some $\mathbf{n}\in\overline{M}_{\ge 0}$.*

*Proof.* Let $\mathbf{m}=x^{m_1}y^{m_2}z^{m_3}$ be a Laurent monomial of weight $j$. To prove (i), assume that $0\le j<r-a$. This means that

$$0\le \frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3-\lfloor\frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3\rfloor<\frac{r-a}{r}.$$

Thus $\phi_2(y\cdot\mathbf{m})=\phi_2(x^{m_1}y^{m_2+1}z^{m_3})=\phi_2(x^{m_1}y^{m_2}z^{m_3})=\phi_2(\mathbf{m})$. The assertions (ii) and (iii) can be proved similarly. The definition of $\phi_k$ implies (iv). □

**Lemma 3.10.** *For each $k=1,2,3$, let $\phi_k$ be the round down function of the weighted blowup with weight $(1,a,r-a)$. Let $\mathbf{k}$ be a lattice point in the monomial lattice $M_k$ and $\mathbf{g}$ a monomial of degree 1 in $M_k$. There exist a monomial $\mathbf{f}\in\{x,y,z\}$ of degree 1 and $\mathbf{m}\in\overline{M}$ such that*

$$\phi_k(\mathbf{f}\cdot\mathbf{m})=\mathbf{g}\cdot\mathbf{k}$$

*satisfying $\phi_k(\mathbf{m})=\mathbf{k}$.*

*Proof.* Here we prove the assertion for the left round down function. Let $\xi, \eta, \zeta$ denote the eigencoordinates for the $G_2$-action. Let $\mathbf{k}$ be a monomial in $M_2$ and $\mathbf{g} \in \{\xi, \eta, \zeta\}$.

Consider the case where $\mathbf{g} = \zeta$. Since $\phi_2$ is surjective, there exists $\mathbf{m} = x^{m_1} y^{m_2} z^{m_3} \in \overline{M}$ such that $\phi_2(\mathbf{m}) = \mathbf{k}$. If $\zeta \cdot \mathbf{k} = \phi_2(z \cdot \mathbf{m})$, then we are done.

Suppose $\zeta \cdot \mathbf{k} \neq \phi_2(z \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 + \frac{r-a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor + 1.$$

There is a positive integer $l_0$[3] such that $\phi_2(\frac{\mathbf{m}}{y^l}) = \mathbf{k}$ for all $0 \leq l \leq l_0$ with $\phi_2\Big(\frac{\mathbf{m}}{y^{l_0+1}}\Big) \neq \mathbf{k}$. Since $\phi_2(z \cdot \frac{\mathbf{m}}{y^{l_0}}) = \zeta \cdot \mathbf{k}$, the assertion follows.

For the other cases, we can prove the assertion similarly. □

3.3. **Economic resolutions.** By the fact that the quotient variety $X = \mathbb{C}^3/G$ has terminal singularities, $X$ does not admit crepant resolutions. However $X$ has a certain toric resolution introduced by Danilov [5] (see also [17]).

**Definition 3.11.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1, a, r-a)$. For each $1 \leq i < r$, let $v_i := \frac{1}{r}(i, \overline{ai}, \overline{r-ai}) \in L$ where $^{-}$ denotes the residue modulo $r$. The *economic resolution* of $\mathbb{C}^3/G$ is the toric variety obtained by the consecutive weighted blowups at $v_1, v_2, \ldots, v_{r-1}$ from $\mathbb{C}^3/G$.

**Proposition 3.12** (see [17])**.** *Let $\varphi\colon Y \to X = \mathbb{C}^3/G$ be the economic resolution of $\mathbb{C}^3/G$. For each $1 \leq i < r$, let $E_i$ denote the exceptional divisor of $\varphi$ corresponding to the lattice point $v_i$.*

(i) *The toric variety $Y$ is smooth and projective over $X$.*
(ii) *The morphism $\varphi$ satisfies*

$$K_Y = \varphi^*(K_X) + \sum_{1 \leq i < r} \frac{i}{r} E_i.$$

*In particular, each discrepancy is $0 < \frac{i}{r} < 1$.*

From the fan of $Y$, we can see that $Y$ can be covered by three open sets $U_2$, $U_3$ and $U_1$, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r-a)$, respectively. Note that $U_2$ and $U_3$ are isomorphic to the economic resolutions for the singularity of type $\frac{1}{a}(1, -r, r)$ and of type $\frac{1}{r-a}(1, r, -r)$, respectively.

---

[3]This integer $l_0$ is the largest integer satisfying

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r}l \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

**Remark 3.13.** Let $\Sigma$ be the toric fan of the economic resolution $Y$. Note that the number of 3-dimensional cones in $\Sigma$ is $2r-1$ and that the number of 3-dimensional cones containing $e_1$ is $r$. $\Diamond$

**Example 3.14.** Let $G$ be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.10. The fan of the economic resolution of the quotient variety $\mathbb{C}^3/G$ is shown in Figure 3.3.

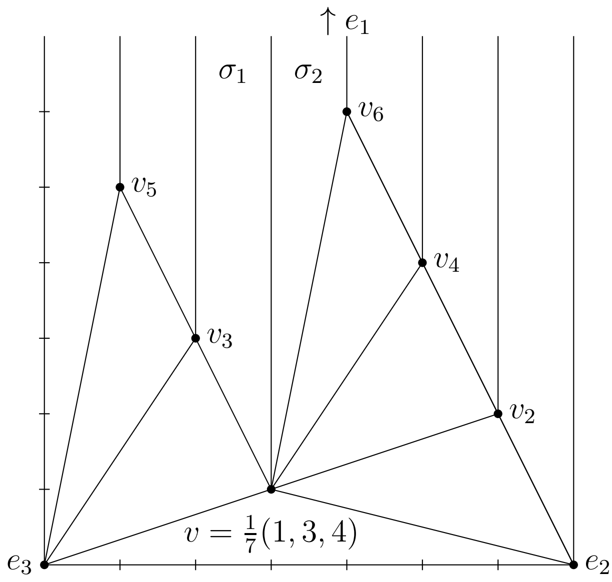


FIGURE 3.3. Fan of the economic resolution for $\frac{1}{7}(1,3,4)$

The toric variety corresponding to the fan consisting of the cones on the left side of $v = \frac{1}{7}(1,3,4)$ is the economic resolution of the quotient $\frac{1}{3}(1,2,1)$. On the other hand, the toric variety corresponding to the fan consisting of the cones on the right side of $v$ is the economic resolution of the quotient $\frac{1}{4}(1,3,1)$. $\Diamond$

## 4. Moduli interpretations of economic resolutions

This section contains our main theorem. First, we explain how to find a set $\mathfrak{S}(r,a)$ of $G$-bricks using the round down functions and a recursion process. In Section 4.3, we show that there exists a stability parameter $\theta$ such that every $G$-brick in $\mathfrak{S}(r,a)$ is $\theta$-stable.

4.1. ***G*-bricks and stability parameters for $\frac{1}{r}(1,r-1,1)$.** Let $G$ be the group of $\frac{1}{r}(1,r-1,1)$ type, i.e. $a=1$ or $r-1$. In this case, Kędzierski [10] proved that $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of $\mathbb{C}^3/G$.

**Theorem 4.1** (Kędzierski [10]). *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1,a,r-a)$ with $a=1$ or $r-1$. Then $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$.*

For each $1 \le i < r$, set $v_i = \frac{1}{r}(i, r-i, i)$. Set $v_0 = e_2$ and $v_r = e_3$. The toric fan corresponding to $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ consists of the following $2r-1$ maximal cones and their faces:

$$\begin{aligned} \sigma_i &= \mathrm{Cone}(e_1, v_{i-1}, v_i) && \text{for } 1 \le i \le r, \\ \sigma_{r+i} &= \mathrm{Cone}(e_3, v_{i-1}, v_i) && \text{for } 1 \le i \le r-1. \end{aligned}$$

Each 3-dimensional cone has a corresponding (Nakamura's) $G$-graph:

$$\begin{aligned} \Gamma_i &= \{1, y, y^2, \dots, y^{i-1}, z, z^2, \dots, z^{r-i}\} && \text{for } 1 \le i \le r, \\ \Gamma_{r+i} &= \{1, y, y^2, \dots, y^{i-1}, x, x^2, \dots, x^{r-i}\} && \text{for } 1 \le i \le r-1, \end{aligned} \tag{4.2}$$

with $S(\Gamma_j) = \sigma_j^\vee \cap M$. As the cone $\sigma_j$ is 3-dimensional, the $G$-prebrick $\Gamma_j$ is a $G$-brick. Furthermore, $U(\Gamma_j) = D(\Gamma_j) \cong \mathbb{C}^3$.

By Ito-Nakajima [7], all $G$-bricks in (4.2) are $\theta$-stable for any $\theta \in \Theta_+$ where

$$\Theta_+ := \left\{ \theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for } \rho \ne \rho_0 \right\}. \tag{4.3}$$

**Example 4.4.** Let $G$ be the finite group of type $\frac{1}{3}(1,2,1)$ with eigencoordinates $\xi, \eta, \zeta$. Set $v_1 = \frac{1}{3}(1,2,1)$ and $v_2 = \frac{1}{3}(2,1,2)$. Recall that the economic resolution $Y$ of $X = \mathbb{C}^3/G$ can be obtained by the sequence of the weighted blowups:

$$Y \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} X,$$

where $\varphi_1$ is the weighted blowup with weight $(1,2,1)$ and $\varphi_2$ is the toric morphism induced by the weighted blowup with weight $(2,1,2)$. The fan corresponding to $Y$ consists of the following five 3-dimensional cones and their faces:

$$\begin{aligned} &\sigma_1 = \mathrm{Cone}(e_1, e_2, v_1), \quad \sigma_2 = \mathrm{Cone}(e_1, v_1, v_2), \quad \sigma_3 = \mathrm{Cone}(e_1, v_2, e_3), \\ &\sigma_4 = \mathrm{Cone}(e_3, e_2, v_1), \quad \sigma_5 = \mathrm{Cone}(e_3, v_1, v_2). \end{aligned}$$

The following

$$\begin{aligned} &\Gamma_1 = \{1, \zeta, \zeta^2\}, \quad \Gamma_2 = \{1, \eta, \zeta\}, \quad \Gamma_3 = \{1, \eta, \eta^2\}, \\ &\Gamma_4 = \{1, \xi, \xi^2\}, \quad \Gamma_5 = \{1, \xi, \eta\}. \end{aligned}$$

are their corresponding $G$-bricks. $\Diamond$

4.2. **$G$-bricks for $\frac{1}{r}(1,a,r-a)$.** In this section, we assign a $G$-brick $\Gamma_\sigma$ with $S(\Gamma_\sigma) = \sigma^\vee \cap M$ to each maximal cone $\sigma$ in the fan of the economic resolution $Y$.

Let $G$ be the group of type $\frac{1}{r}(1,a,r-a)$ with $r$ coprime to $a$, $X$ the quotient $\mathbb{C}^3/G$, and $\varphi\colon Y \to X$ the economic resolution of $X$. Then $Y$ can be covered by $U_2$, $U_3$ and $U_1$, which are the unions of the affine

toric varieties corresponding to the cones on the left side of, the right side of, and below the lattice point $v = \frac{1}{r}(1, a, r-a)$, respectively.

**Proposition 4.5.** *For $k = 1, 2, 3$, let $\Gamma'$ be a $G_k$-brick. Define*

$$\Gamma := \left\{ \mathbf{m} \in \overline{M} \,\middle|\, \phi_k(\mathbf{m}) \in \Gamma' \right\}.$$

*The set $\Gamma$ is a $G$-brick with $S(\Gamma) = S(\Gamma')$.*

*Proof.* Since $\phi_k(\mathbf{1}) = \mathbf{1} \in \Gamma'$, we have $\mathbf{1} \in \Gamma$. To show that $\Gamma$ satisfies (ii) in Definition 2.8, we need to show that there exists a unique monomial of weight $\rho$ in $\Gamma$ for each $\rho \in G^\vee$. Fix a positive integer $i$ such that the weight of $x^i$ is $\rho$. Consider the monomial $\phi_k(x^i)$ in $M_k$ and its weight $\chi \in G_k^\vee$. Since $\Gamma'$ is a $G_k$-brick, there exists a unique element $\mathbf{k}_\chi$ of weight $\chi$. Since the $G_k$-invariant monomial $\frac{\mathbf{k}_\chi}{\phi_k(x^i)}$ is in the lattice $M$, it follows from Lemma 3.7 that

$$\phi_k \colon x^i \cdot \left( \frac{\mathbf{k}_\chi}{\phi_2(x^i)} \right) \mapsto \mathbf{k}_\chi,$$

i.e. $x^i \cdot \left( \frac{\mathbf{k}_\chi}{\phi_2(x^i)} \right)$ is in $\Gamma$. To show uniqueness, assume that two monomials $\mathbf{m}, \mathbf{m}'$ of the same weight are mapped into $\Gamma'$. As $\phi_k(\mathbf{m})$ and $\phi_k(\mathbf{m}')$ are of the same weight, we have $\phi_k(\mathbf{m}) = \phi_k(\mathbf{m}') \in \Gamma'$. From Lemma 3.7,

$$\phi_k(\mathbf{m}) = \phi_k \left( \mathbf{m}' \cdot \frac{\mathbf{m}}{\mathbf{m}'} \right) = \phi_k(\mathbf{m}') \cdot \frac{\mathbf{m}}{\mathbf{m}'},$$

and hence $\mathbf{m} = \mathbf{m}'$. From Lemma 3.10, it follows that $\Gamma$ is connected as $\Gamma'$ is connected.

For (iii) in Definition 2.8, assume that $\mathbf{n}' \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{n}, \mathbf{n}' \in \overline{M}_{\geq 0}$. We need to show $\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) \in \Gamma'$. From

$$\phi_k(\mathbf{n}' \cdot \mathbf{n} \cdot \mathbf{m}_\rho) = \frac{\phi_k(\mathbf{n}' \cdot \mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)} \cdot \frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \cdot \phi_k(\mathbf{m}_\rho) \in \Gamma',$$

it follows that $\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) = \frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \cdot \phi_k(\mathbf{m}_\rho)$ is in $\Gamma'$ because $\Gamma'$ is a $G_k$-prebrick. This proves that $\Gamma$ is a $G$-prebrick.

It remains to prove that $S(\Gamma) = S(\Gamma')$. Note that $S(\Gamma)$ is generated by $\frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)}$ for $\mathbf{n} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$. Let $\mathbf{n}$ be a genuine monomial in $\overline{M}_{\geq 0}$ and $\mathbf{m}_\rho$ an element in $\Gamma$. Let $\mathbf{k}_\chi$ denote $\phi_k(\mathbf{m}_\rho) \in \Gamma'$. Define $\mathbf{k}$ to be $\frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$. From the definition of the round down functions, we know that $\mathbf{k}$ is a genuine monomial in $\xi, \eta, \zeta$. Since $\frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)}$ is $G$-invariant, it follows that

$$\frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)} = \frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k\big(\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)\big)} = \frac{\frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \cdot \phi_k(\mathbf{m}_\rho)}{\phi_k\big(\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)\big)} = \frac{\mathbf{k} \cdot \mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k} \cdot \mathbf{k}_\chi)}$$

from Lemma 3.7. This proves $S(\Gamma) \subset S(\Gamma')$.

For the opposite inclusion, by Lemma 2.11, it suffices to show that $\frac{\mathbf{g} \cdot \mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{g} \cdot \mathbf{k}_\chi)}$ is in $S(\Gamma)$ for all $\mathbf{g} \in \{\xi, \eta, \zeta\}$ and $\mathbf{k}_\chi \in \Gamma'$. By Lemma 3.10

there are $\mathbf{n} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$ such that $\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho) = \mathbf{g} \cdot \mathbf{k}_\chi$. Lemma 4.6 implies that $\operatorname{wt}_{\Gamma'}(\mathbf{g} \cdot \mathbf{k}_\chi) = \phi_k\big(\operatorname{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)\big)$. Thus

$$\frac{\mathbf{g} \cdot \mathbf{k}_\chi}{\operatorname{wt}_{\Gamma'}(\mathbf{g} \cdot \mathbf{k}_\chi)} = \frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\operatorname{wt}_{\Gamma'}\big(\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)\big)} = \frac{\phi_k(\mathbf{n} \cdot \mathbf{m}_\rho)}{\phi_k\big(\operatorname{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)\big)} = \frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\operatorname{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)},$$

and we proved the proposition. □

**Lemma 4.6.** *With the notation as in Proposition 4.5, if* $\mathbf{m} \in \overline{M}$, *then*

$$\operatorname{wt}_{\Gamma'}\big(\phi_k(\mathbf{m})\big) = \phi_k\big(\operatorname{wt}_\Gamma(\mathbf{m})\big).$$

*Proof.* Since $\phi_k(\mathbf{m})$ is of the same weight as $\phi_k\big(\operatorname{wt}_\Gamma(\mathbf{m})\big)$ by Lemma 3.7, the assertion follows from the fact that $\phi_k\big(\operatorname{wt}_\Gamma(\mathbf{m})\big) \in \Gamma'$. □

**Lemma 4.7.** *With the notation as in Proposition 4.5, let* $\mathbf{m}$ *be the Laurent monomial of weight* $j$ *in* $\Gamma = \big\{\mathbf{m} \in \overline{M} \,\big|\, \phi_k(\mathbf{m}) \in \Gamma'\big\}$.

(i) *If* $k = 2$ *and* $0 \leq j < r - a$, *then* $\phi_2(y \cdot \mathbf{m}) \in \Gamma$.
(ii) *If* $k = 3$ *and* $0 \leq j < a$, *then* $\phi_3(z \cdot \mathbf{m}) \in \Gamma$.
(iii) *If* $k = 1$ *and* $0 \leq j < r - 1$, *then* $\phi_1(x \cdot \mathbf{m}) \in \Gamma$.

*Proof.* Lemma 3.9 implies the assertion. □

**Proposition 4.8.** *Let* $G$ *be the group of type* $\frac{1}{r}(1, a, r-a)$ *with* $r$ *coprime to* $a$. *Let* $\Sigma_{\max}$ *be the set of maximal cones in the fan of the economic resolution* $Y$ *of* $X = \mathbb{C}^3/G$. *Then there exists a set* $\mathfrak{S}(r, a)$ *of* $G$*-bricks such that there is a bijective map* $\Sigma_{\max} \to \mathfrak{S}(r, a)$ *sending* $\sigma$ *to* $\Gamma_\sigma$ *satisfying* $S(\Gamma_\sigma) = \sigma^\vee \cap M$. *In particular,* $U(\Gamma_\sigma)$ *is isomorphic to the smooth toric variety* $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ *corresponding to* $\sigma$.

*Proof.* From Section 4.1, the assertion holds when $a = 1$ or $r - 1$. We use induction on $r$ and $a$.

Let $\sigma$ be a 3-dimensional cone in the fan of the economic resolution $Y$ of $X = \mathbb{C}^3/G$. For $v = \frac{1}{r}(1, a, r-a)$, we have three cases:

(1) the cone $\sigma$ is below the vector $v$.
(2) the cone $\sigma$ is on the left side of the vector $v$.
(3) the cone $\sigma$ is on the right side of the vector $v$.

**Case (1) the cone $\sigma$ is below the vector $v$.** Since there is a unique 3-dimensional cone below $v$, $\sigma = \operatorname{Cone}(v, e_2, e_3)$. Consider the central round down function $\phi_1$ of the weighted blowup with weight $(1, a, r-a)$. For $\mathbf{m} = x^{m_1} y^{m_2} z^{m_3} \in \overline{M}$, note that

$$\phi_1(\mathbf{m}) = \mathbf{1} \quad \text{if and only if} \quad m_2 = m_3 = 0 \text{ and } 0 \leq \tfrac{m_1}{r} < 1.$$

The set $\Gamma := \phi_1^{-1}(\mathbf{1}) = \{1, x, x^2, \ldots, x^{r-1}\}$ is a $G$-prebrick with the property $S(\Gamma) = \sigma^\vee \cap M$. Since the corresponding cone $\sigma(\Gamma)$ is equal to $\sigma$, the $G$-prebrick $\Gamma$ is a $G$-brick.

**Case (2) the cone $\sigma$ is on the left side of $v$.** From the fan of the economic resolution, it follows that $U_2$ is isomorphic to the economic resolution $Y_2$ of $\frac{1}{a}(1,-r,r)$ with eigencoordinates $\xi,\eta,\zeta$. There exists a unique 3-dimensional cone $\sigma'$ in the toric fan of $Y_2$ corresponding to $\sigma$. Let $G_2$ be the group of type $\frac{1}{a}(1,-r,r)$. Note that $a$ is strictly less than $r$ so that we can use induction.

Assume that there exists a $G_2$-brick $\Gamma'$ with $S(\Gamma')=(\sigma')^\vee\cap M$. By Proposition 4.5, there is a $G$-brick $\Gamma$ with $S(\Gamma)=S(\Gamma')=\sigma^\vee\cap M$.

**Case (3) the cone $\sigma$ is on the right side of $v$.** The case where the cone $\sigma$ is on the right side of $v$ can be proved similarly. □

**Definition 4.9.** A $G$-brick $\Gamma$ in $\mathfrak{S}(r,a)$ described above is called a *Danilov $G$-brick.*

**Proposition 4.10.** *With the notation as is in Proposition 4.5, we have $D(\Gamma')\cong D(\Gamma)$. Moreover we have a commutative diagram*

$$\begin{array}{ccc} U(\Gamma') & \xrightarrow{\cong} & U(\Gamma) \\ \big\downarrow & & \big\downarrow \\ D(\Gamma') & \xrightarrow{\cong} & D(\Gamma) \end{array}$$

*with the vertical morphisms closed embeddings. Therefore for a $G$-brick $\Gamma\in\mathfrak{S}(r,a)$, we have $U(\Gamma)=D(\Gamma)\cong\mathbb{C}^3$.*

*Proof.* Let $\Gamma$ be a $G$-brick and $\Gamma'$ the corresponding $G_k$-brick. Let $\xi,\eta,\zeta$ denote the eigencoordinate for the $G_k$-action. From (2.19), the coordinate rings of the affine schemes $D(\Gamma)$, $D(\Gamma')$ are

$$\mathbb{C}[D(\Gamma)]=\mathbb{C}[x_\rho,y_\rho,z_\rho \,|\, \rho\in G^\vee]/I_\Gamma,$$
$$\mathbb{C}[D(\Gamma')]=\mathbb{C}[\xi_\chi,\eta_\chi,\zeta_\chi \,|\, \chi\in G_k^\vee]/I_{\Gamma'}$$

where the ideal $I_\Gamma$ is $\langle$the quadrics in (2.18), $\mathbf{f}_\rho-1 \,|\, \mathbf{f}_\rho\in\Lambda(\Gamma)\rangle$ and the ideal $I_{\Gamma'}$ is $\langle$the commutative relations, $\mathbf{g}_\chi-1 \,|\, \mathbf{g}_\chi\in\Lambda(\Gamma')\rangle$.

By Lemma 3.10, we have an algebra epimorphism

$$\mu\colon \mathbb{C}[x_\rho,y_\rho,z_\rho \,|\, \rho\in G^\vee]\to\mathbb{C}[D(\Gamma')] \quad \mathbf{f}_\rho\mapsto\mathbf{k}_{(\chi)}$$

defined as follows on the $3r$ generators $\mathbf{f}_\rho\in\{x_\rho,y_\rho,z_\rho\}$. Let $\mathbf{m}_\rho$ be the unique element of weight $\rho$ in $\Gamma$ and $\chi$ the weight of $\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)$. Then $\mathbf{k}:=\frac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$ is a monomial, so $\mathbf{k}$ induces a linear map $\mathbf{k}_{(\chi)}$ on the vector space $\mathbb{C}\cdot\phi_k(\mathbf{m}_\rho)$. Then $\mu$ is the morphism sending $\mathbf{f}_\rho$ to $\mathbf{k}_{(\chi)}$. Since the generators of $I_\Gamma$ are in $\ker\mu$, $\mu$ induces an epimorphism $\overline{\mu}\colon\mathbb{C}[D(\Gamma)]\to\mathbb{C}[D(\Gamma')]$.

To construct the inverse of $\overline{\mu}$, first we show that if $\mu(\mathbf{f}_\rho)=\mu(\mathbf{f}'_{\rho'})$, then $\mathbf{f}_\rho\equiv\mathbf{f}'_{\rho'} \mod I_\Gamma$. If $\mu(\mathbf{f}_\rho)=\mu(\mathbf{f}'_{\rho'})$, then $\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)=\phi_k(\mathbf{f}'\cdot\mathbf{m}_{\rho'})$ and $\phi_k(\mathbf{m}_\rho)=\phi_k(\mathbf{m}_{\rho'})$. Since both $\mathbf{f}_\rho$ and $\mathbf{f}'_{\rho'}$ are degree of 1, $\mathbf{f}=\mathbf{f}'$. By (iv) in Lemma 3.9, we may assume that $\mathbf{m}_{\rho'}=\mathbf{n}\cdot\mathbf{m}_\rho$ for some $\mathbf{n}\in\overline{M}_{\geq 0}$. Since $\phi_k(\mathbf{m}_\rho)=\phi_k(\mathbf{m}_{\rho'})$, $\mathbf{n}$ induces a linear map equal to

1 on $\mathbf{m}_\rho$, i.e. $\mathbf{m}_{(\rho)} \equiv 1 \mod I_\Gamma$ because $\mathbf{m}_{\rho'} = \mathbf{n}\cdot\mathbf{m}_\rho$ is a base. From the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}\cdot\mathbf{m}_\rho & \xrightarrow{\cdot\mathbf{n}} & \mathbb{C}\cdot\mathbf{m}_{\rho'} \\ \Big\downarrow \mathbf{f}_\rho & & \Big\downarrow \mathbf{f}_{\rho'} \\ \mathbb{C}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho) & \xrightarrow{\cdot\mathbf{n}} & \mathbb{C}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'}), \end{array}$$

it suffices to show that $\mathbf{n}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)$ is a base, which implies that $\mathbf{n}$ induces a linear map equal to 1 on $\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)$. Since

$$\begin{aligned} \phi_k(\mathbf{n}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)) &= \phi_k\Big(\mathbf{f}\cdot\mathbf{m}_{\rho'}\cdot\frac{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)}{\mathbf{f}\cdot\mathbf{m}_\rho}\Big) \\ &= \phi_k(\mathbf{f}\cdot\mathbf{m}_{\rho'})\cdot\frac{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)}{\mathbf{f}\cdot\mathbf{m}_\rho} \\ &= \phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)\cdot\frac{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)}{\mathbf{f}\cdot\mathbf{m}_\rho} = \phi_k\big(\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)\big), \end{aligned}$$

the monomial $\mathbf{n}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)$ is in $\Gamma$ and is a base.

Now define the algebra morphism $\nu\colon \mathbb{C}[\xi_\chi,\eta_\chi,\zeta_\chi \,|\, \chi\in G_k^\vee] \to \mathbb{C}[D(\Gamma)]$ by $\nu(\mathbf{g}_\chi) = \mathbf{f}_\rho$ for $\mathbf{g}_\chi \in \{\xi_\chi,\eta_\chi,\zeta_\chi\}$ so that $\mu(\mathbf{f}_\rho) = \mathbf{g}_\chi$. Since the generators of $I_{\Gamma'}$ are in $\ker\nu$, $\nu$ induces $\overline{\nu}\colon \mathbb{C}[D(\Gamma')] \to \mathbb{C}[D(\Gamma)]$.

To show $\overline{\nu}$ is surjective, we prove that generators $\mathbf{f}_\rho$ are in the image of $\nu$. For $\mathbf{f}_\rho$ such that $\frac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$ is of degree $\le 1$, $\mathbf{f}_\rho$ is in the image of $\nu$ by definition. By Lemma 4.11 below, it follows that $\overline{\nu}$ is surjective.

Since $\overline{\mu}$ and $\overline{\nu}$ are the inverses of each other, $D(\Gamma)$ is isomorphic to $D(\Gamma')$. Note that $U(\Gamma) = D(\Gamma) \cong \mathbb{C}^3$ for $\Gamma \in \mathfrak{S}(r,1)$ from Section 4.1. Using induction, we get $D(\Gamma)\cong\mathbb{C}^3$ for $\Gamma\in\mathfrak{S}(r,a)$. ☐

**Lemma 4.11.** *In the situation as in Proposition 4.10, define*

$$S := \Big\{\mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\} \,\Big|\, \tfrac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \text{ is of degree} \le 1\Big\}.$$

*If $\frac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$ is of degree $\ge 2$ for some $\mathbf{f}_\rho \in \{x_\rho,y_\rho,z_\rho\}$, then $\mathbf{f}_\rho$ can be written as a multiple of some elements in $S$ modulo $I_\Gamma$.*

*Proof.* We prove this for the left round down function $\phi_2$. Note that $\frac{\phi_k(y\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$ is of degree $\le 1$ for all $\rho\in G^\vee$. Thus $y_\rho$'s are in $S$.

Suppose that $\frac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$ is of degree $\ge 2$ with $\mathbf{m}_\rho = x^{m_1}y^{m_2}z^{m_3}$. Then the monomial $\mathbf{f}$ is either $x$ or $z$. In the case where $\mathbf{f} = z$, this means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor \ge \frac{a}{r}.$$

As in the proof of Lemma 3.10, there is a positive integer $l$ such that $\phi_2(\frac{\mathbf{m}_\rho}{y^{l'}}) = \phi_2(\mathbf{m}_\rho)$ for all $0\le l'\le l$ with $\phi_2\big(\frac{\mathbf{m}_\rho}{y^{l+1}}\big) \ne \phi_2(\mathbf{m}_\rho)$. Note

that $\frac{\phi_2(\mathbf{f}\cdot\mathbf{m}_{\rho'})}{\phi_2(\mathbf{m}_{\rho'})}$ is of degree 1 where $\mathbf{m}_{\rho'} = \frac{\mathbf{m}_\rho}{y^l}$. Thus $\mathbf{f}_{\rho'} \in S$. From the commutation relations

$$\begin{array}{ccc} \mathbb{C}\cdot\mathbf{m}_{\rho'} & \xrightarrow{\cdot y^l} & \mathbb{C}\cdot\mathbf{m}_\rho \\ \Big\downarrow \mathbf{f}_{\rho'} & & \Big\downarrow \mathbf{f}_\rho \\ \mathbb{C}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'}) & \xrightarrow{\cdot y^l} & \mathbb{C}\cdot\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho), \end{array}$$

since $y^l$ induces a linear map on $\mathbb{C}\cdot\mathbf{m}_{\rho'}$ set to be 1, we have

$$\mathbf{f}_\rho \equiv \mathbf{f}_\rho \cdot y^l_{(\rho')} \equiv y^l_{(\rho)} \cdot \mathbf{f}_{\rho'} \mod I_\Gamma.$$

As all $y_\rho$'s are in $S$, the assertion follows. □

**Example 4.12.** Let $G$ be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.10. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.

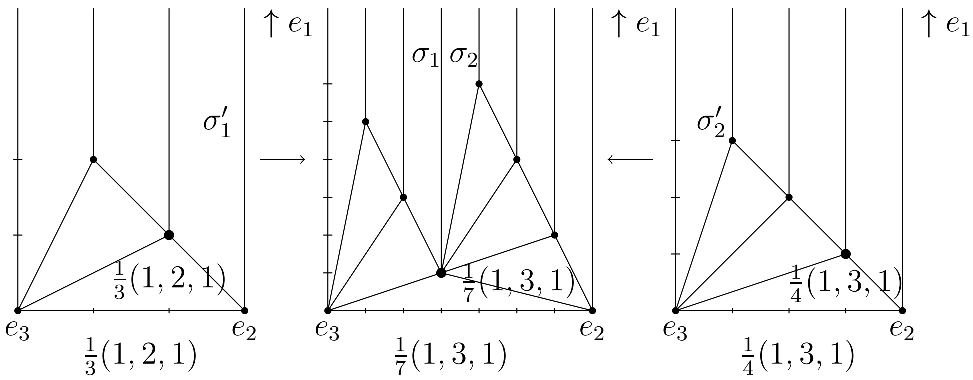


FIGURE 4.1. Recursion process for $\frac{1}{7}(1,3,4)$

We now calculate $G$-bricks associated to the following cones:

$$\sigma_1 := \mathrm{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(3,2,5)\right),$$
$$\sigma_2 := \mathrm{Cone}\left((1,0,0), \tfrac{1}{7}(6,4,3), \tfrac{1}{7}(1,3,4)\right).$$

Note that the left side of the fan corresponds to the economic resolution for the quotient singularity of type $\frac{1}{3}(1,2,1)$, which is $G_2$-Hilb $\mathbb{C}^3$, where $G_2$ is of type $\frac{1}{3}(1,2,1)$. Let $\xi, \eta, \zeta$ denote the eigencoordinates. Let $\sigma_1'$ be the cone in the fan of $G_2$-Hilb $\mathbb{C}^3$ which corresponds to $\sigma_1$. Observe that the corresponding $G_2$-brick is

$$\Gamma_1' = \left\{1, \zeta, \zeta^2\right\}.$$

Since the left round down function $\phi_2$ is

$$\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor}\zeta^{m_3},$$

the $G$-brick corresponding to $\sigma_1$ is

$$\begin{aligned}\Gamma_1 &\stackrel{\text{def}}{=} \left\{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \,\middle|\, \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma_1'\right\}\\ &= \left\{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\}.\end{aligned}$$

On the other hand, the right side of the fan corresponds to the economic resolution of the quotient variety $\frac{1}{4}(1,3,1)$ which is $G_3$-Hilb $\mathbb{C}^3$, where $G_3$ is of type $\frac{1}{4}(1,3,1)$ with eigencoordinates $\alpha, \beta, \gamma$. Let $\sigma_2'$ be the cone in the fan of $G_2$-Hilb $\mathbb{C}^3$ which corresponds to $\sigma_2$. Observe that the corresponding $G_3$-brick is

$$\Gamma_2' = \left\{1, \beta, \beta^2, \beta^3\right\}.$$

Since the right round down function $\phi_3$ is

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \alpha^{m_1}\beta^{m_2}\gamma^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor},$$

the $G$-brick corresponding to $\sigma_2$ is

$$\begin{aligned}\Gamma_2 &\stackrel{\text{def}}{=} \left\{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \,\middle|\, \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma_2'\right\}\\ &= \left\{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z}, \tfrac{y^3}{z^2}\right\}.\end{aligned}$$

From Example 2.13, $\sigma(\Gamma_1) = \sigma_1$ and $\sigma(\Gamma_2) = \sigma_2$. ♢

4.3. **Stability parameters for $\mathfrak{S}(r,a)$.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with $r$ coprime to $a$. We may assume $2a < r$. Let $G_2$ and $G_3$ be the groups of type $\frac{1}{a}(1, -r, r)$ and of type $\frac{1}{r-a}(1, r, -r)$, respectively.

Given stability conditions $\theta^{(2)}$ for Danilov $G_2$-bricks and $\theta^{(3)}$ for Danilov $G_3$-bricks, take a GIT parameter $\theta_P \in \Theta$ satisfying the following system of linear equations:

$$\begin{cases} \theta^{(2)}(\chi) = \sum\limits_{\phi_2(\rho)=\chi} \theta_P(\rho) & \text{for all } \chi \in G_2^\vee, \\ \theta^{(3)}(\chi') = \sum\limits_{\phi_3(\rho)=\chi'} \theta_P(\rho) & \text{for all } \chi' \in G_3^\vee. \end{cases} \tag{4.13}$$

Define the GIT parameter $\psi \in \Theta$ by

$$\psi(\rho) = \begin{cases} -1 & \text{if } 0 \le \mathrm{wt}(\rho) < a\ , \\ 1 & \text{if } r - a \le \mathrm{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Observe that $\sum\limits_{\phi_k(\rho)=\chi} \psi(\rho) = 0$ for all $\chi \in G_k^\vee$ [4]. For a sufficiently large natural number $m$, set

$$\theta := \theta_P + m\psi. \tag{4.15}$$

[4]In addition, if any $\theta \in \Theta$ satisfies that $\sum_{\phi_k(\rho)=\chi} \theta(\rho) = 0$ for all $\chi \in G_k^\vee$ and $k = 2, 3$, then $\theta$ must be a constant multiple of $\psi$. This also explains the existence of a solution $\theta_P$ for (4.13).

We claim that every $\Gamma \in \mathfrak{S}(r,a)$ is $\theta$-stable.

**Example 4.16.** As in Example 4.12, let $G$ be the group of type $\frac{1}{7}(1,3,4)$. For each $0 \leq i \leq 6$, let $\rho_i$ denote the irreducible representation of $G$ whose weight is $i$. We saw that the left side of the fan is $G_2$-Hilb $\mathbb{C}^3$, where $G_2$ is of type $\frac{1}{3}(1,2,1)$ and that the right side of the fan is $G_3$-Hilb $\mathbb{C}^3$, where $G_3$ is of type $\frac{1}{4}(1,3,1)$. Let $\{\chi_0,\chi_1,\chi_2\}$ and $\{\chi_0',\chi_1',\chi_2',\chi_3'\}$ be the characters of $G_2$ and $G_3$, respectively. Take GIT parameters $\theta^{(2)}$, $\theta^{(3)}$ corresponding to $G$-Hilb such as (see (4.3)):

$$\theta^{(2)} = (-2,1,1), \quad \theta^{(3)} = (-3,1,1,1).$$

We have the following system of linear equations:

$$\begin{cases} -2 &= \theta_P(\rho_0) + \theta_P(\rho_3) + \theta_P(\rho_6), \\ 1 &= \theta_P(\rho_1) + \theta_P(\rho_4), \\ 1 &= \theta_P(\rho_2) + \theta_P(\rho_5), \\ -3 &= \theta_P(\rho_0) + \theta_P(\rho_4), \\ 1 &= \theta_P(\rho_1) + \theta_P(\rho_5), \\ 1 &= \theta_P(\rho_2) + \theta_P(\rho_6), \\ 1 &= \theta_P(\rho_3). \end{cases}$$

Take $\theta_P = (-1,3,3,1,-2,-2,-2)$ as a partial solution. For the parameter $\psi = (-1,-1,-1,0,1,1,1)$, define $\theta = \theta_P + m\psi$ for large $m$.

Consider the following $G$-brick

$$\Gamma = \left\{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\}.$$

Let $\mathcal{F}$ be the submodule of $C(\Gamma)$ with basis $A = \left\{z, \frac{z}{y}, \frac{z^2}{y}\right\}$. Note that $\psi(\mathcal{F}) > 0$ and

$$\phi_2^{-1}\big(\phi_2(A)\big) = \left\{z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\} \supsetneq A.$$

Thus $\theta(\mathcal{F})$ is positive for large enough $m$. More precisely,

$$\theta(\mathcal{F}) = 3 - m + (-2+m) + (-2+m) = m - 1$$

is positive if $m > 1$.

On the other hand, consider the submodule $\mathcal{G}$ of $C(\Gamma)$ with basis $B = \left\{\frac{z}{y}, \frac{z^2}{y}\right\}$. Note that $\psi(\mathcal{G}) = 0$ and $\phi_2^{-1}\big(\phi_2(B)\big) = B$. In this case, the set $\phi_2(B)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$ with

$$\theta^{(2)}(\mathcal{G}') = \theta(\mathcal{G}).$$

Since $C(\Gamma')$ is $\theta^{(2)}$-stable, $\theta^{(2)}(\mathcal{G}')$ is positive. Hence $\theta(\mathcal{G})$ is positive. $\Diamond$

**Lemma 4.17.** *Let $\theta$ be the parameter in* (4.15). *For the set $\mathfrak{S}(r,a)$ in Proposition 4.8, if $\Gamma$ is in $\mathfrak{S}(r,a)$, then $\Gamma$ is $\theta$-stable.*

*Proof.* Let $\Gamma$ be a $G$-brick in $\mathfrak{S}$ and $\sigma$ the cone corresponding to $\Gamma$. We have the following three cases as in Section 4.2:

(1) the cone $\sigma$ is below the vector $v$.

(2) the cone $\sigma$ is on the left side of the vector $v$.
(3) the cone $\sigma$ is on the right side of the vector $v$.

In Case (1), $\Gamma = \{1, x, x^2, \ldots, x^{r-2}, x^{r-1}\}$. By Lemma 2.14, any nonzero proper submodule $\mathcal{G}$ of $C(\Gamma)$ is given by

$$A = \{x^j, x^{j+1}, \ldots, x^{r-2}, x^{r-1}\}$$

for some $1 \leq j \leq r-1$. Since $\psi(\mathcal{G}) > 0$, $\Gamma$ is $\theta$-stable for sufficiently large $m$.

We now consider Case (2). Let $\Gamma'$ be the $G_2$-brick corresponding to $\Gamma$. Let $\mathcal{G}$ be a submodule of $C(\Gamma)$ with $\mathbb{C}$-basis $A \subset \Gamma$. Lemma 3.9 and Lemma 2.14 imply that if $\mathbf{m}_\rho \in A$ for $0 \leq \mathrm{wt}(\mathbf{m}_\rho) < a$, then $\phi_2^{-1}\big(\phi_2(\mathbf{m}_\rho)\big) \subset A$. Thus $\psi(\mathcal{G}) \geq 0$ from the definition of $\psi$.

If $\psi(\mathcal{G}) > 0$, then it follows that $\theta(\mathcal{G}) > 0$ for sufficiently large $m$.

Let us assume that $\psi(\mathcal{G}) = 0$. Note that $A = \phi_2^{-1}\big(\phi_2(A)\big)$; otherwise there exists $\mathbf{m}_\rho$ in $\phi_2^{-1}\big(\phi_2(A)\big) \setminus A$ with $0 \leq \mathrm{wt}(\mathbf{m}_\rho) < a$. To show that $\theta(\mathcal{G})$ is positive, we prove that $\phi_2(A)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$ and that $\theta(\mathcal{G}) = \theta^{(2)}(\mathcal{G}')$. Since $\theta$ satisfies the equations (4.13), it suffices to show that $\phi_2(A)$ gives a submodule of $C(\Gamma')$. Let $\xi, \eta, \zeta$ be the coordinates of $\mathbb{C}^3$ with respect to the action of $G_2$. By Lemma 2.14, it is enough to show that if $\mathbf{g} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$ for some $\mathbf{g} \in \{\xi, \eta, \zeta\}$ and $\mathbf{m}_\rho \in A$, then $\mathbf{g} \cdot \phi_2(\mathbf{m}_\rho) \in \phi_2(A)$. Suppose that $\mathbf{g} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$ for some $\mathbf{m}_\rho \in A$. By Lemma 3.10, there exists $\mathbf{m}_{\rho'}$ such that

$$\phi_2(\mathbf{f} \cdot \mathbf{m}_{\rho'}) = \mathbf{g} \cdot \phi_2(\mathbf{m}_\rho)$$

with $\phi_2(\mathbf{m}_{\rho'}) = \phi_2(\mathbf{m}_\rho)$ for some $\mathbf{f} \in \{x, y, z\}$. In particular, $\mathbf{f} \cdot \mathbf{m}_{\rho'}$ is in $\Gamma$. Since $A = \phi_2^{-1}\big(\phi_2(A)\big)$, we have $\mathbf{m}_{\rho'} \in A$, which implies $\mathbf{f} \cdot \mathbf{m}_{\rho'} \in A$ as $A$ is a $\mathbb{C}$-basis of $\mathcal{G}$. Thus $\mathbf{g} \cdot \phi_2(\mathbf{m}_\rho)$ is in $\phi_2(A)$. $\square$

**Remark 4.18.** At this moment, our stability parameter $\theta$ in (4.15) has nothing to do with *Kędzierski's GIT chamber* $\mathfrak{C}(r, a)$ described in [11]. In Section 5, it is shown that the parameter $\theta$ is in $\mathfrak{C}(r, a)$. $\diamondsuit$

### 4.4. **Main Theorem.**

**Theorem 4.19.** *The economic resolution $Y$ of a 3-fold terminal quotient singularity $X = \mathbb{C}^3/G$ is isomorphic to the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.*

*Proof.* From Proposition 4.8 and Lemma 4.17, Proposition 2.21 implies that there exists an open immersion from $Y$ to $Y_\theta$ fitting in the following commutative diagram:

$$\begin{array}{ccc} Y & \to & Y_\theta \\ & \searrow & \downarrow \\ & & X. \end{array}$$

Since both $Y$ and $Y_\theta$ are projective over $X$, the open immersion $Y \to Y_\theta$ is a closed embedding. As both $Y$ and $Y_\theta$ are 3-dimensional and irreducible, this embedding is an isomorphism. □

**Conjecture 4.20.** *The moduli space $\mathcal{M}_\theta$ is irreducible.*

Proposition 4.10 implies that the irreducible component $Y_\theta$ is actually a connected component. In addition, if every torus invariant $\theta$-stable $G$-constellation lies over the birational component $Y_\theta$, then $\mathcal{M}_\theta$ is irreducible. For $a = 2$, we can prove Conjecture 4.20 so the economic resolution is isomorphic to $\mathcal{M}_\theta$ for $\theta \in \mathfrak{S}(r, a)$ (See [8]). We hope to establish this more generally in future work.

**Remark 4.21.** By construction, $\mathcal{M}_0 = \operatorname{Spec} \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ is the moduli space of 0-semistable $G$-constellations up to $S$-equivalence. Since there exists an algebra isomorphism $\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)} \to \mathbb{C}[x, y, z]^G$, $\mathcal{M}_0$ is isomorphic to $\mathbb{C}^3/G$. In particular, $\mathcal{M}_0$ is irreducible. ♢

## 5. Kędzierski's GIT chamber

Kędzierski [11] described his GIT cone in $\Theta$ using a set of inequalities. Using his lemma, we can prove further that the cone is actually a GIT chamber $\mathfrak{C}$. In this section, we provide a description of $\mathfrak{C}$ using the $A_{r-1}$ root system. Define

$$\mathfrak{S}(r, a)_0 = \{\Gamma \in \mathfrak{S}(r, a) \,\big|\, x \notin \Gamma\}.$$

**Kędzierski's lemma.** By the same argument as in Lemma 6.7 of [11], we can prove that it suffices to check the $\theta$-stability for $G$-bricks $\Gamma$ not containing $x$.

**Lemma 5.1** (Kędzierski's lemma [11])**.** *For a parameter $\theta \in \Theta$, the following are equivalent.*

(i) *Every $\Gamma \in \mathfrak{S}(r, a)$ is $\theta$-stable.*
(ii) *Every $\Gamma \in \mathfrak{S}(r, a)_0$ is $\theta$-stable.*

Let $A$ be the finite group of type $\frac{1}{r}(a, r-a)$. Since $A \cong G$ as groups, the GIT parameter space $\Theta$ of $G$-constellations can be canonically identified with that of $A$-constellations.

Since $G$-constellations which $x$ acts trivially on are supported on the hyperplane $(x = 0) \subset \mathbb{C}^3$, they can be considered as $A$-constellations. As $\Gamma \in \mathfrak{S}(r, a)_0$ is the set of $G$-bricks corresponding to $G$-constellations supported on $(x = 0) \subset \mathbb{C}^3$, Lemma 5.1 implies that the GIT chamber for $\mathfrak{S}(r, a)$ is equal to a GIT chamber of $A$-constellations.

**Kędzierski's GIT chamber.** We describe a set of simple roots $\Delta$ so that $Y_\theta$ is isomorphic to the economic resolution for $\theta \in \mathfrak{C}(\Delta)$. After considering the case of $a = 1$, we describe simple roots for the case of $\frac{1}{r}(1, a, r-a)$ using a recursion process.

*Root system $A_{r-1}$.* Identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$. Let $\{\varepsilon_i \,|\, i \in I\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. Define

$$\Phi := \{\varepsilon_i - \varepsilon_j \,|\, i, j \in I, i \neq j\}.$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated by $\Phi$. Elements in $\Phi$ are called *roots.*

For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. Let $\rho_i$ denote the irreducible representation of $G$ of weight $i$. Note that each root $\alpha$ can be considered as the support of a submodule of a $G$-constellation. In other words, $\alpha_i$ corresponds to the dimension vector of $\rho_i$. In general we consider a root $\alpha = \sum_i n_i \alpha_i$ as the dimension vector of the representation $\oplus n_i \rho_i$. Abusing notation, let $\alpha = \sum_i n_i \alpha_i$ denote the corresponding representation $\oplus n_i \rho_i$.

Let $\Delta$ be a set of simple roots. Define $\mathfrak{C}(\Delta) \subset \Theta$ associated to $\Delta$ as

$$\mathfrak{C}(\Delta) := \{\theta \in \Theta \,|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta\}.$$

Note that for the cone $\Theta_+$ for $G$-Hilb in (4.3), the corresponding set of simple roots is

$$\Delta_+ = \{\varepsilon_i - \varepsilon_{i-a} \in \Phi \,|\, i \in I, i \neq 0\} = \{\alpha_i \,|\, i \in I, i \neq 0\}.$$

*The case of $\frac{1}{r}(1, r-1, 1)$.* From Theorem 4.1, we know that the economic resolution of $X = \mathbb{C}^3/G$ is isomorphic to $G\text{-Hilb}\,\mathbb{C}^3$ if $G$ is of type $\frac{1}{r}(1, r-1, 1)$. Thus in this case, the $G$-bricks are just Nakamura's $G$-graphs, which are $\theta$-stable for $\theta \in \Theta_+$, where

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for } \rho \neq \rho_0\right\}.$$

The corresponding set of simple roots is

$$\begin{aligned}\Delta &= \left\{\varepsilon_i - \varepsilon_{i+1} \in \Phi \,\middle|\, i \in I, i \neq 0\right\} \\ &= \left\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{r-1} - \varepsilon_0\right\}.\end{aligned}$$

**Example 5.2.** For the group of type $\frac{1}{3}(1, 2, 1)$, let $\left\{\varepsilon_j^L \,\middle|\, j = 0, 1, 2\right\}$ be the standard basis of $\mathbb{Q}^3$. The corresponding set of simple roots is

$$\Delta^L = \left\{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\right\}.$$

Similarly, for the group of type $\frac{1}{4}(1, 3, 1)$ with $\left\{\varepsilon_k^R \,\middle|\, k = 0, 1, 2, 3\right\}$ the standard basis of $\mathbb{Q}^4$,

$$\Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}$$

is the corresponding set of simple roots for type $\frac{1}{4}(1, 3, 1)$. $\lozenge$

*The case of* $\frac{1}{r}(1,a,r-a)$. Let $G$ be the group of type $\frac{1}{r}(1,a,r-a)$. Let $\Delta^L$ and $\Delta^R$ denote the sets of simple roots for the types of $\frac{1}{a}(1,-r,r)$ and of $\frac{1}{r-a}(1,r,-r)$, respectively. As in Section 5, let

$$\{\varepsilon^L_l \mid l=0,1,\dots,a-1\}, \quad \{\varepsilon^R_k \mid k=0,1,\dots,r-a-1\}$$

be the standard basis of $\mathbb{Q}^a$ and $\mathbb{Q}^{r-a}$, respectively.

From the two sets $\Delta^L$ and $\Delta^R$, we construct a set $\Delta$ of simple roots in $A_{r-1}$ as follows. First, as in Section 5, let the standard basis $\{\varepsilon_i \mid i\in I\}$ of $\mathbb{Q}^r$ be identified with the union of the two sets

$$\{\varepsilon^L_l \mid l=0,1,\dots,a-1\} \text{ and } \{\varepsilon^R_k \mid k=0,1,\dots,r-a-1\}$$

using the following identification:

$$\begin{cases} \varepsilon^L_l &= \varepsilon_i \quad \text{with } i\equiv l \mod a \qquad\quad \text{if } r-a\le i<r, \\ \varepsilon^R_k &= \varepsilon_i \quad \text{with } i\equiv k \mod (r-a) \quad \text{if } 0\le i<r-a. \end{cases} \tag{5.3}$$

With the identification above, define $\Delta$ to be

$$\Delta = \Delta^L \cup \{\varepsilon_{\lfloor\frac{r-1}{a}\rfloor a} - \varepsilon_{\lfloor\frac{r-1}{r-a}\rfloor (r-a)-a}\} \cup \Delta^R. \tag{5.4}$$

The root $\varepsilon_{\lfloor\frac{r-1}{a}\rfloor a} - \varepsilon_{\lfloor\frac{r-1}{r-a}\rfloor (r-a)-a}$ is called the *added root* in $\Delta$. Note that $\Delta$ is actually a set of simple roots in $A_{r-1}$.

**Definition 5.5.** With $\Delta$ as above, the corresponding Weyl chamber

$$\mathfrak{C}(r,a) := \mathfrak{C}(\Delta) = \{\theta\in\Theta \mid \theta(\alpha)>0 \quad \forall\alpha\in\Delta\}$$

is called *Kędzierski's GIT chamber* for $G=\frac{1}{r}(1,a,r-a)$.

**Proposition 5.6.** *Let $\mathfrak{C}(r,a)$ be Kędzierski's GIT chamber.*

(i) *The parameter $\psi$ in* (4.14) *is a ray of $\mathfrak{C}(r,a)$.*
(ii) *Any $G$-brick in $\mathfrak{S}(r,a)$ is $\theta$-stable for $\theta\in\mathfrak{C}(r,a)$.*
(iii) *The cone $\mathfrak{C}(r,a)$ is a full GIT chamber.*

*Proof.* We may assume $a<r-a$. First, by construction, $\psi$ is zero on the sets $\Delta^L$ and $\Delta^R$ with the identification (5.4). To prove (i), it remains to show that $\psi(\alpha)$ is positive where $\alpha$ is the added root in $\Delta$. Since

$$\alpha = \varepsilon_{\lfloor\frac{r-1}{a}\rfloor a} - \varepsilon_{\lfloor\frac{r-1}{r-a}\rfloor (r-a)-a} = \sum_{\phi_2(\rho_i)=\chi_0} \alpha_i + \alpha_{r-a},$$

where $\chi_0$ is the trivial representation of $G_2$, (i) follows.

For $\theta$ defined by (4.15), every $\Gamma\in\mathfrak{S}(r,a)_0$ is $\theta$-stable. For the group $A$ of type $\frac{1}{r}(a,-a)$, Kronheimer [13] showed that the chamber structure of the GIT parameter space of $A$-constellations is the same as the Weyl chamber structure of $A_{r-1}$[5]. Thus for $\mathfrak{S}(r,a)_0$ considered as $A$-constellations, we have a Weyl chamber of the $A_{r-1}$ root system containing the parameter $\theta$.

By Kędzierski's lemma, to prove (ii), it suffices to show that $\mathfrak{C}(r,a)$ contains the parameter $\theta$. Observe that every parameter in $\mathfrak{C}(r,a)$

[5]For an explicit description, see Section 5.1 in [8]

satisfies the system of equations (4.13) for some $\theta^{(2)} \in \mathfrak{C}(a, -r)$ and $\theta^{(3)} \in \mathfrak{C}(r-a, r)$ by construction. Since $\psi$ in (4.14) is a ray of the chamber $\mathfrak{C}(r, a)$, it follows that $\theta \in \mathfrak{C}(r, a)$.

It remains to prove (iii). By considering $G$-constellations supported on the hyperplane $(x = 0) \subset \mathbb{C}^3$, it follows that any facet of $\mathfrak{C}(r, a)$ is an actual GIT wall in $\Theta$. Therefore Kędzierski's GIT chamber $\mathfrak{C}(r, a)$ is a full GIT chamber in the stability parameter space $\Theta$ (see [8,9]). $\square$

**Proposition 5.7.** *Assume that $a < r - a$. Let $\theta$ be an element in $\mathfrak{C}(r, a)$. Then $\theta(\alpha_i)$ is negative if and only if $0 \le i < a$. Thus any $\theta$-stable $G$-constellation is generated by $\rho_0, \rho_1, \dots, \rho_{a-1}$.*

*Proof.* Let $\Delta$ be the set of simple roots corresponding to $\mathfrak{C}(r, a)$. Recall that any positive sum of simple roots is positive on $\theta$.

Suppose that $0 \le i < a$. From the identification (5.3), note that $\varepsilon_i$ is identified with $\varepsilon^R_k$ for some $k$ and that $\varepsilon_{i-a} = \varepsilon_{i+(r-a)}$ is identified with $\varepsilon^L_l$ for some $l$. Note that $\varepsilon_{\lfloor \frac{r-1}{a} \rfloor a}$ is identified with a vector $\varepsilon^L$ and that $\varepsilon_{\lfloor \frac{r-1}{r-a} \rfloor (r-a)-a}$ is identified with a vector $\varepsilon^R$. Since we added the root $\varepsilon_{\lfloor \frac{r-1}{a} \rfloor a} - \varepsilon_{\lfloor \frac{r-1}{r-a} \rfloor (r-a)-a}$ to $\Delta$, the root $\alpha_i = \varepsilon_i - \varepsilon_{i-a} = \varepsilon^R_k - \varepsilon^L_l$ is a negative sum of simple roots in $\Delta$.

Suppose that $a \le i < r - a$. The root $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$ is a sum of simple roots in $\Delta^R$. A recursive argument yields that $\alpha_i$ is a positive sum of simple roots in $\Delta^R$. Thus $\alpha_i$ is a positive sum of simple roots in $\Delta$.

Consider the case where $r - a \le i < r$ and the root $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. From the identification (5.3), $\varepsilon_i$ is identified with $\varepsilon^L_k$ for some $k$ and $\varepsilon_{i-a}$ is identified with $\varepsilon^R_l$ for some $l$. Thus $\alpha_i = \varepsilon^L_k - \varepsilon^R_l$ is a positive sum of simple roots in $\Delta$ with the same reason as the case where $0 \le i < a$. $\square$

**Example 5.8.** Let $G$ be the group of type $\frac{1}{7}(1, 3, 4)$. From the fan of the economic resolution of this case (see Example 3.14), the left and right sides are the economic resolutions of singularities of $\frac{1}{3}(1, 2, 1)$ and $\frac{1}{4}(1, 3, 1)$, respectively. By Example 5.2, we have two sets

$$\Delta^L = \{\varepsilon^L_1 - \varepsilon^L_2, \varepsilon^L_2 - \varepsilon^L_0\} \text{ and } \Delta^R = \{\varepsilon^R_1 - \varepsilon^R_2, \varepsilon^R_2 - \varepsilon^R_3, \varepsilon^R_3 - \varepsilon^R_0\}.$$

As in the construction (5.4), the corresponding set of simple roots is

$$\begin{aligned}\Delta &= \{\varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \underline{\varepsilon_6 - \varepsilon_1}, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_0\} \\ &= \{\alpha_4 + \alpha_1, \alpha_5 + \alpha_2, \underline{-\alpha_1 - \alpha_5 - \alpha_2}, \alpha_1 + \alpha_5, \alpha_2 + \alpha_6, \alpha_3\},\end{aligned}$$

where the underlined root is the added root as in (5.4). Thus the set of parameters $\theta \in \Theta$ satisfying

$$\begin{array}{lll} \theta(\rho_4 \oplus \rho_1) > 0, & \theta(\rho_5 \oplus \rho_2) > 0, & \theta(\rho_1 \oplus \rho_5 \oplus \rho_2) < 0, \\ \theta(\rho_1 \oplus \rho_5) > 0, & \theta(\rho_2 \oplus \rho_6) > 0, & \theta(\rho_3) > 0 \end{array}$$

is Kędzierski's GIT chamber $\mathfrak{C}(r, a)$ where $\rho_i$ is the irreducible representation of $G$ of weight $i$.

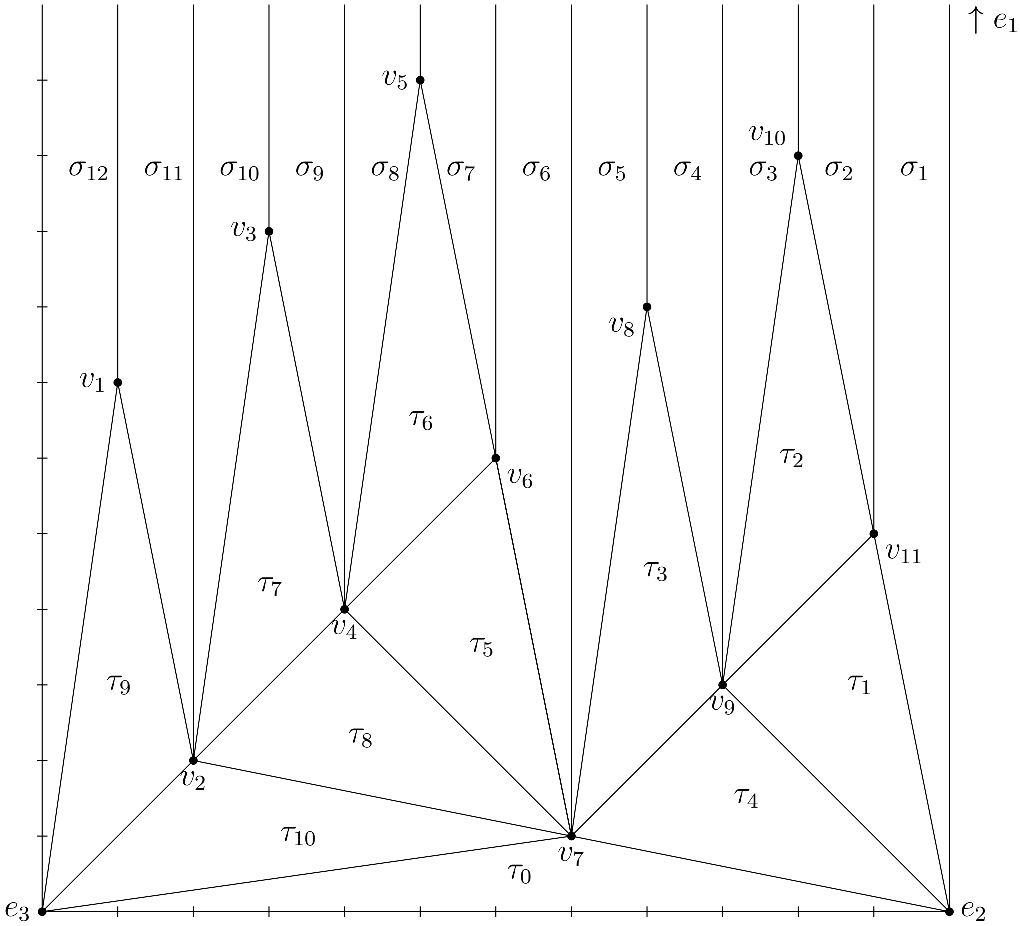


FIGURE 6.1. Toric fan of the economic resolution for $\frac{1}{12}(1,7,5)$

The rays of the chamber $\mathfrak{C}(r,a)$ are the row vectors of the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

with the dual basis $\{\theta_i\}$ with respect to $\{\rho_i\}$. Observe that for any $\theta \in \mathfrak{C}(r,a)$, $\theta(\rho_i)$ is negative if and only if $0 \leq i < 3$. $\Diamond$

## 6. EXAMPLE: TYPE $\frac{1}{12}(1,7,5)$

In this section, as a concrete example, we calculate Danilov $G$-bricks and the corresponding set of simple roots $\Delta$ for the group $G$ of type $\frac{1}{12}(1,7,5)$.

Let $G$ be the finite group of type $\frac{1}{12}(1,7,5)$ with eigencoordinates $x, y, z$ and $L$ the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{12}(1,7,5)$. Let $X$ denote the quotient variety $\mathbb{C}^3/G$ and $Y$ the economic resolution of $X$. The toric fan $\Sigma$ of $Y$ is shown in Figure 6.1.

To use the recursion process in Section 4, first we need to investigate the cases of type $\frac{1}{7}(1,2,5)$ and of type $\frac{1}{5}(1,2,3)$. Let $G_2$ be the group of type $\frac{1}{7}(1,2,5)$ with eigencoordinates $\xi_2, \eta_2, \zeta_2$ and $G_3$ be the group of type $\frac{1}{5}(1,2,3)$ with eigencoordinates $\xi_3, \eta_3, \zeta_3$. Consider the toric fans $\Sigma_2$ and $\Sigma_3$ of the economic resolutions for the type $\frac{1}{7}(1,2,5)$ and the type $\frac{1}{5}(1,2,3)$, respectively.

***G*-bricks.** We now calculate $G$-bricks corresponding to the following two maximal cones in $\Sigma$:

$$\begin{aligned} \sigma_4 &= \mathrm{Cone}\left(\tfrac{1}{12}(12,0,0), \tfrac{1}{12}(3,9,3), \tfrac{1}{12}(8,8,4)\right), \\ \tau_3 &= \mathrm{Cone}\left(\tfrac{1}{12}(1,7,5), \tfrac{1}{12}(3,9,3), \tfrac{1}{12}(8,8,4)\right). \end{aligned}$$

The cones $\sigma_4, \tau_3$ are on the right side of the lowest vector $v = \frac{1}{12}(1,7,5)$. Their corresponding cones $\sigma_4', \tau_3'$ in $\Sigma_3$, respectively, are

$$\begin{aligned} \sigma_4' &= \mathrm{Cone}\left(\tfrac{1}{5}(5,0,0), \tfrac{1}{5}(1,2,3), \tfrac{1}{5}(1,1,4)\right), \\ \tau_3' &= \mathrm{Cone}\left(\tfrac{1}{5}(0,0,5), \tfrac{1}{5}(1,2,3), \tfrac{1}{5}(1,1,4)\right). \end{aligned} \tag{6.1}$$

Observe that the cones $\sigma_4', \tau_3'$ are on the left side of $\Sigma_3$. To use the recursion, let $G_{32}$ be the group of type $\frac{1}{2}(1,1,1)$ with eigencoordinates $\xi_{32}, \eta_{32}, \zeta_{32}$. Let $\Sigma_{32}$ denote the fan of the economic resolution of the quotient $\mathbb{C}^3/G_{32}$. In $\Sigma_{32}$, there exist two cones $\sigma_4'', \tau_3''$ corresponding to $\sigma_4', \tau_3'$, respectively:

$$\begin{aligned} \sigma_4'' &= \mathrm{Cone}\left(\tfrac{1}{2}(2,0,0), \tfrac{1}{2}(0,2,0), \tfrac{1}{2}(1,1,1)\right), \\ \tau_3'' &= \mathrm{Cone}\left(\tfrac{1}{2}(0,0,2), \tfrac{1}{2}(0,2,0), \tfrac{1}{2}(1,1,1)\right). \end{aligned}$$

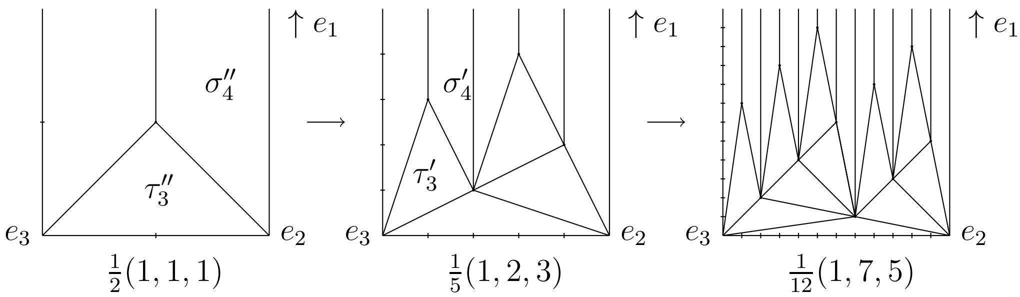


FIGURE 6.2. Recursion process for $\frac{1}{12}(1,7,5)$

As in Section 4.1, the $G_{32}$-bricks $\Gamma_4'', \Gamma_3''$ corresponding to $\sigma_4'', \tau_3''$ are

$$\begin{aligned} \Gamma_4'' &= \{1, \zeta_{23}\}, \\ \Gamma_3'' &= \{1, \xi_{23}\}. \end{aligned}$$

Using the left round down function $\phi_{32}$ for $\frac{1}{5}(1,2,3)$

$$\phi_{32}\colon \xi_3^a\eta_3^b\zeta_3^c \;\mapsto\; \xi_{32}^a\eta_{32}^{\lfloor\frac{a+2b+3c}{5}\rfloor}\zeta_{32}^c,$$

we can see that the $G_3$-bricks $\Gamma'_4, \Gamma'_3$ corresponding to $\sigma'_4, \tau'_3$ are

$$\begin{aligned}
\Gamma'_4 &\overset{\text{def}}{=} \phi_{32}^{-1}\big(\Gamma''_4\big) &&= \big\{1,\ \eta_3,\ \eta_3^2,\ \zeta_3,\ \tfrac{\zeta_3}{\eta_3}\big\},\\
\Gamma'_3 &\overset{\text{def}}{=} \phi_{32}^{-1}\big(\Gamma''_2\big) &&= \big\{1,\ \eta_3,\ \eta_3^2,\ \xi_3,\ \xi_3\eta_3\big\}.
\end{aligned}$$

To get the $G$-bricks $\Gamma_4$ and $\Gamma_3$ corresponding to $\sigma_4$ and $\tau_3$, respectively, we use the right round down function $\phi_3$ for $\frac{1}{12}(1,7,5)$:

$$\phi_3\colon x^ay^bz^c \;\mapsto\; \xi_3^a\eta_3^b\zeta_3^{\lfloor\frac{a+7b+5c}{12}\rfloor}.$$

We get

$$\begin{aligned}
\Gamma_4 &\overset{\text{def}}{=} \phi_3^{-1}(\Gamma'_4) &&= \left\{1, y, \tfrac{y}{z}, \tfrac{y^2}{z}, \tfrac{y^2}{z^2}, z, z^2, z^3, z^4, \tfrac{z^4}{y}, \tfrac{z^5}{y}, \tfrac{z^6}{y}\right\},\\
\Gamma_3 &\overset{\text{def}}{=} \phi_3^{-1}(\Gamma'_2) &&= \left\{1, x, xz, xz^2, xy, \tfrac{xy}{z}, y, \tfrac{y}{z}, \tfrac{y^2}{z}, \tfrac{y^2}{z^2}, z, z^2\right\}.
\end{aligned}$$

Let us consider the following two cones in $\Sigma$:

$$\begin{aligned}
\sigma_9 &= \mathrm{Cone}\left(\tfrac{1}{12}(12,0,0), \tfrac{1}{12}(9,3,9), \tfrac{1}{12}(4,4,8)\right),\\
\tau_7 &= \mathrm{Cone}\left(\tfrac{1}{12}(2,2,10), \tfrac{1}{12}(9,3,9), \tfrac{1}{12}(4,4,8)\right).
\end{aligned}$$

Observe that the cones $\sigma_9$, $\tau_7$ are on the left side of $v$. The cones in $\Sigma_2$ corresponding to $\sigma_9$, $\tau_7$ are

$$\begin{aligned}
\sigma'_9 &= \mathrm{Cone}\left(\tfrac{1}{7}(12,0,0), \tfrac{1}{7}(5,3,4), \tfrac{1}{7}(2,4,3)\right),\\
\tau'_7 &= \mathrm{Cone}\left(\tfrac{1}{7}(1,2,5), \tfrac{1}{7}(5,3,4), \tfrac{1}{7}(2,4,3)\right).
\end{aligned}$$

Note that the cones $\sigma'_9$, $\tau'_7$ are on the right side of the fan $\Sigma_2$ and that the right side is equal to the fan $\Sigma_3$ of the economic resolution for $\frac{1}{5}(1,2,3)$. Moreover, the cones in $\Sigma_3$ corresponding to $\sigma'_9$, $\tau'_7$ are $\sigma'_4$, $\tau'_3$, respectively, in (6.1). Thus the corresponding $G_{23}$-bricks $\Gamma''_9$, $\Gamma''_7$ are:

$$\begin{aligned}
\Gamma''_9 &= \big\{1,\ \eta_{23},\ \eta_{23}^2,\ \zeta_{23},\ \tfrac{\zeta_{23}}{\eta_{23}}\big\},\\
\Gamma''_7 &= \big\{1,\ \xi_{23},\ \xi_{23}\eta_{23},\ \eta_{23},\ \eta_{23}^2\big\},
\end{aligned}$$

where $G_{23}$ denotes the group of type $\frac{1}{5}(1,2,3)$ with eigencoordinates $\xi_{23}, \eta_{23}, \zeta_{23}$. Using the right round down function $\phi_{23}$ for $\frac{1}{7}(1,2,5)$

$$\phi_{23}\colon \xi_2^a\eta_2^b\zeta_2^c \;\mapsto\; \xi_{23}^a\eta_{23}^b\zeta_{23}^{\lfloor\frac{a+2b+5c}{7}\rfloor},$$

we can calculate the $G_2$-bricks corresponding to $\sigma'_9, \tau'_7$:

$$\begin{aligned}
\Gamma'_9 &\overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma''_9) &&= \left\{1,\ \eta_2,\ \eta_2^2,\ \zeta_2,\ \zeta_2^2,\ \tfrac{\zeta_2^2}{\eta_2},\ \tfrac{\zeta_2^3}{\eta_2}\right\},\\
\Gamma'_7 &\overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma''_7) &&= \left\{1,\ \xi_2,\ \xi_2\eta_2,\ \xi_2\zeta_2,\ \eta_2,\ \eta_2^2,\ \zeta_2,\ \zeta_2^2\right\}.
\end{aligned}$$

Lastly, from the left round down function $\phi_2$ for $\frac{1}{12}(1,7,5)$

$$\phi_2\colon x^ay^bz^c \;\mapsto\; \xi_2^a\eta_2^{\lfloor\frac{a+7b+5c}{12}\rfloor}\zeta_2^c,$$

it follows that the $G$-bricks $\Gamma_9, \Gamma_7$ corresponding to $\sigma_9, \tau_7$ are:

$$\Gamma_9 = \Big\{1, y, y^2, y^3, y^4, y^5, z, z^2, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}, \tfrac{z^2}{y^3}, \tfrac{z^3}{y^3}\Big\},$$
$$\Gamma_7 = \Big\{1, x, xy, xy^2, xy^3, xz, y, y^2, y^3, y^4, y^5, z\Big\}.$$

For $0 \le i \le 12$, let $v_i$ denote the lattice point $\frac{1}{12}(\overline{7i}, i, 12-i)$ in $L$. For each 3-dimensional cone $\sigma$ in Figure 6.1 on page 32, Table 6.1 on page 36 shows the corresponding $G$-brick $\Gamma_\sigma$.

**Kędzierski's GIT chamber.** We calculate Kędzierski's GIT chamber for $\frac{1}{12}(1,7,5)$. Since the economic resolution is $G$-Hilb for the group of type $\frac{1}{r}(1, r-1, 1)$, the sets of simple roots for $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$ are $\{\varepsilon_1-\varepsilon_0\}, \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_0\}$, respectively. By the identification (5.3), the set of simple roots for $\frac{1}{5}(1,2,3)$ is

$$\{\varepsilon_3-\varepsilon_4,\ \underline{\varepsilon_4-\varepsilon_1},\ \varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_0\},$$

where the underlined root is the added root as in (5.4). Similarly, the admissible set of simple roots for $\frac{1}{7}(1,2,5)$ is

$$\{\varepsilon_5-\varepsilon_6,\ \underline{\varepsilon_6-\varepsilon_3},\ \varepsilon_3-\varepsilon_4,\ \varepsilon_4-\varepsilon_1,\ \varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_0\}.$$

Thus the corresponding set of simple roots for $\frac{1}{12}(1,7,5)$ is

$$\left\{\begin{matrix}\varepsilon_5-\varepsilon_6,\ \varepsilon_6-\varepsilon_{10},\ \varepsilon_{10}-\varepsilon_{11},\ \varepsilon_{11}-\varepsilon_8,\ \varepsilon_8-\varepsilon_9,\ \varepsilon_9-\varepsilon_7,\\ \underline{\varepsilon_7-\varepsilon_3},\ \varepsilon_3-\varepsilon_4,\ \varepsilon_4-\varepsilon_1,\ \varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_0\end{matrix}\right\}.$$

With the dual basis $\{\theta_i\}$ with respect to $\{\rho_i\}$, the row vectors of the following matrix are the rays of the admissible Weyl chamber $\mathfrak{C}(r,a)$:

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
-1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 1\\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\
-1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1\\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

| Cone | Generators | $G$-brick $\Gamma_\sigma$ | Coordinates on $U_\sigma$ |
|---|---|---|---|
| $\sigma_1$ | $e_1, e_2, v_{11}$ | $1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}$ | $\frac{x}{z^5}, \frac{y}{z^{11}}, z^{12}$ |
| $\sigma_2$ | $e_1, v_{10}, v_{11}$ | $1, y, \frac{y}{z}, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9$ | $\frac{x}{z^5}, \frac{y^2}{z^{10}}, \frac{z^{11}}{y}$ |
| $\sigma_3$ | $e_1, v_9, v_{10}$ | $1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, \frac{y^2}{z^3}, \frac{y^2}{z^4}, \frac{y^2}{z^5}, z, z^2, z^3, z^4$ | $\frac{xz^5}{y^2}, \frac{y^3}{z^9}, \frac{z^{10}}{y^2}$ |
| $\sigma_4$ | $e_1, v_8, v_9$ | $1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, z, z^2, z^3, z^4, \frac{z^4}{y}, \frac{z^5}{y}, \frac{z^6}{y}$ | $\frac{xy}{z^4}, \frac{y^4}{z^8}, \frac{z^9}{y^3}$ |
| $\sigma_5$ | $e_1, v_7, v_8$ | $1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, \frac{y^3}{z^2}, \frac{y^3}{z^3}, \frac{y^3}{z^4}, \frac{y^4}{z^4}, \frac{y^4}{z^5}, z, z^2$ | $\frac{xz^4}{y^3}, \frac{y^5}{z^7}, \frac{z^8}{y^4}$ |
| $\sigma_6$ | $e_1, v_6, v_7$ | $1, y, z, z^2, \frac{z^2}{y}, \frac{z^3}{y}, \frac{z^3}{y^2}, \frac{z^4}{y^2}, \frac{z^5}{y^2}, \frac{z^5}{y^3}, \frac{z^6}{y^3}, \frac{z^6}{y^4}$ | $\frac{xy^2}{z^3}, \frac{y^6}{z^6}, \frac{z^7}{y^5}$ |
| $\sigma_7$ | $e_1, v_5, v_6$ | $1, y, y^2, y^3, z, z^2, \frac{z^2}{y}, \frac{z^3}{y}, \frac{z^3}{y^2}, \frac{z^4}{y^2}, \frac{z^5}{y^2}, \frac{z^5}{y^3}$ | $\frac{xy^2}{z^3}, \frac{y^7}{z^5}, \frac{z^6}{y^6}$ |
| $\sigma_8$ | $e_1, v_4, v_5$ | $1, y, y^2, y^3, y^4, y^5, \frac{y^5}{z}, \frac{y^5}{z^2}, \frac{y^6}{z^2}, z, z^2, \frac{z^2}{y}$ | $\frac{xz^2}{y^5}, \frac{y^8}{z^4}, \frac{z^5}{y^7}$ |
| $\sigma_9$ | $e_1, v_3, v_4$ | $1, y, y^2, y^3, y^4, y^5, z, z^2, \frac{z^2}{y}, \frac{z^2}{y^2}, \frac{z^2}{y^3}, \frac{z^3}{y^3}$ | $\frac{xy^3}{z^2}, \frac{y^9}{z^3}, \frac{z^4}{y^8}$ |
| $\sigma_{10}$ | $e_1, v_2, v_3$ | $1, y, y^2, y^3, y^4, y^5, y^6, \frac{y^6}{z}, \frac{y^7}{z}, \frac{y^8}{z}, \frac{y^9}{z}, z$ | $\frac{xz}{y^6}, \frac{y^{10}}{z^2}, \frac{z^3}{y^9}$ |
| $\sigma_{11}$ | $e_1, v_1, v_2$ | $1, y, y^2, y^3, y^4, y^5, y^6, z, \frac{z}{y}, \frac{z}{y^2}, \frac{z}{y^3}, \frac{z}{y^4}$ | $\frac{xy^4}{z}, \frac{y^{11}}{z^1}, \frac{z^2}{y^{10}}$ |
| $\sigma_{12}$ | $e_1, e_3, v_1$ | $1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$ | $\frac{x}{y^7}, y^{12}, \frac{z}{y^{11}}$ |
| $\tau_1$ | $e_2, v_9, v_{11}$ | $1, x, xz, xz^2, xz^3, xz^4, x^2, x^2z, z, z^2, z^3, z^4$ | $\frac{x^3}{z^3}, \frac{y}{x^2z}, \frac{z^5}{x}$ |
| $\tau_2$ | $v_9, v_{10}, v_{11}$ | $1, x, z, xz, z^2, xz^2, z^3, xz^3, z^4, xz^4, y, \frac{y}{z}$ | $\frac{x^2z}{y}, \frac{y^2}{xz^5}, \frac{z^5}{x}$ |
| $\tau_3$ | $v_7, v_8, v_9$ | $1, x, xy, \frac{xy}{z}, xz, xz^2, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, z, z^2$ | $\frac{x^2z}{y}, \frac{y^3}{xz^4}, \frac{z^4}{xy}$ |
| $\tau_4$ | $e_2, v_7, v_9$ | $1, x, x^2, x^3, x^4, xz, xz^2, x^2z, x^3z, x^4z, z, z^2$ | $\frac{x^5}{z}, \frac{y}{x^2z}, \frac{z^3}{x^3}$ |
| $\tau_5$ | $v_4, v_6, v_7$ | $1, x, xy, xz, xz^2, \frac{xz^2}{y}, x^2, x^2y, y, z, z^2, \frac{z^2}{y}$ | $\frac{x^3y}{z^2}, \frac{y^2}{x^2}, \frac{z^3}{xy}$ |
| $\tau_6$ | $v_4, v_5, v_6$ | $1, x, xy, xz, xz^2, \frac{xz^2}{y}, y, y^2, y^3, z, z^2, \frac{z^2}{y}$ | $\frac{x^2}{y^2}, \frac{y^5}{xz^2}, \frac{z^3}{xy^2}$ |
| $\tau_7$ | $v_2, v_3, v_4$ | $1, x, xy, xy^2, xy^3, xz, y, y^2, y^3, y^4, y^5, z$ | $\frac{x^2}{y^2}, \frac{y^6}{xz}, \frac{z^2}{xy^3}$ |
| $\tau_8$ | $v_2, v_4, v_7$ | $1, x, xy, xz, x^2, x^2y, x^3, x^3y, x^4, x^4y, y, z$ | $\frac{x^5}{z}, \frac{y^2}{x^2}, \frac{z^2}{x^3y}$ |
| $\tau_9$ | $e_3, v_1, v_2$ | $1, x, xy, xy^2, xy^3, xy^4, y, y^2, y^3, y^4, y^5, y^6$ | $\frac{x^2}{y^2}, \frac{y^7}{x}, \frac{z}{xy^4}$ |
| $\tau_{10}$ | $e_3, v_2, v_7$ | $1, x, xy, x^2, x^2y, x^3, x^3y, x^4, x^4y, x^5, x^6, y$ | $\frac{x^7}{y}, \frac{y^2}{x^2}, \frac{z}{x^5}$ |
| $\tau_0$ | $e_2, e_3, v_7$ | $1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}$ | $x^{12}, \frac{y}{x^7}, \frac{z}{x^5}$ |

TABLE 6.1. $G$-bricks for $G = \frac{1}{12}(1, 7, 5)$

## References

[1] T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.

[2] A. Craw, A. Ishii, *Flops of* $G$-Hilb *and equivalences of derived categories by variation of GIT quotient*, Duke Math. J. **124** (2004), no. 2, 259–307.

[3] A. Craw, D. Maclagan, R. R. Thomas, *Moduli of McKay quiver representations I: The coherent component*, Proc. Lond. Math. Soc. (3) **95** (2007), no. 1, 179–198.

[4] A. Craw, D. Maclagan, R. R. Thomas, *Moduli of McKay quiver representations II: Gröbner basis techniques*, J. Algebra **316** (2007), no. 2, 514–535

[5] V. Danilov, *Birational geometry of three-dimensional toric varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 971–982, 1135; English translation: Math. USSR-Izv. **21** (1983), no. 2, 269–280.

[6] S. Davis, T. Logvinenko, M. Reid, *How to calculate* $A$-Hilb $\mathbb{C}^n$ *for* $\frac{1}{r}(a, b, 1, \ldots, 1)$, preprint.

[7] Y. Ito, H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Topology **39** (2000), no. 6, 1155–1191.

[8] S.-J. Jung, *McKay Quivers and Terminal Quotient Singularities in Dimension 3*, PhD thesis, University of Warwick, 2014.

[9] S.-J. Jung, *Note on GIT chambers for $G$-constellations*, in preparation.

[10] O. Kędzierski, *Cohomology of the G-Hilbert scheme for* $\frac{1}{r}(1, 1, r-1)$, Serdica Math. J. **30** (2004), no. 2–3, 293–302.

[11] O. Kędzierski, *Danilov's resolution and representations of the McKay quiver*, Tohoku Math. J. (2) **66** (2014), no. 3, 355–375.

[12] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser.(2) **45** (1994), no. 180, 515–530.

[13] P. Kronheimer, *The construction of ALE spaces as a hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683.

[14] D. Morrison, G. Stevens, *Terminal quotient singularities in dimension 3 and 4*, Proc. Amer. Math Soc. **90** (1984), 15–20.

[15] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**. Springer-Verlag, Berlin, 1994.

[16] I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic. Geom. **10** (2001), no.4, 757–779.

[17] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

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